Signal Space

for EE511

by

A. DAHIMENE

Institut de Génie Electrique et Electronique

Université M'Hamed Bougara
Boumerdes

Signal space 3

Signal space

The notion of signal space is fundamental in communication. It allows a simple geometric approach to complex communication problems.

1.1 Review on vector space

Vector spaces form an algebraic structure based on Fields. A vector space is composed of a set V of elements called vectors. This set possesses a composition law called "addition of vector". The set of vectors along with the addition of vectors forms an abelian group. In addition, there exists a multiplication operation between elements (called "scalars") from a field F and vectors from giving vectors.

In this set of notes, we are going to use bold letters to designate vectors and normal letters for scalars. To recapitulate, we must have:

Given $V = \{\mathbf{x}_1, \mathbf{x}_2, \ldots\}$, the elements must satisfy:

$\forall \mathbf{x}, \mathbf{y}, \mathbf{z} \in V \qquad (\mathbf{x}+\mathbf{y})+\mathbf{z} = \mathbf{x}+(\mathbf{y}+\mathbf{z})$ {associativity}

$\exists$ unique $\mathbf{0} \in V \;|\; \mathbf{x}+\mathbf{0} = \mathbf{0}+\mathbf{x} = \mathbf{x}$ {additive identity}

$\forall \mathbf{x} \in V\,;\, \exists -\mathbf{x} \in V \,|\, \mathbf{x}+(-\mathbf{x}) = (-\mathbf{x})+\mathbf{x} = \mathbf{0}$ {additive inverse}

$\forall \mathbf{x}, \mathbf{y} \in V \qquad \mathbf{x}+\mathbf{y} = \mathbf{y}+\mathbf{x}$ {commutativity}

and

$\forall \alpha \in F$ and $\forall \mathbf{x} \in V \qquad \alpha\mathbf{x} \in F$ such that

$\forall \alpha, \beta \in F$ and $\forall \mathbf{x} \in V \qquad (\alpha+\beta)\mathbf{x} = \alpha\mathbf{x}+\beta\mathbf{x}$

$\forall \alpha \in F$ and $\forall \mathbf{x}, \mathbf{y} \in V \qquad \alpha(\mathbf{x}+\mathbf{y}) = \alpha\mathbf{x}+\alpha\mathbf{y}$

We say that we have a vector space V over the field F.

We can define many different types of vector spaces.

Example:

♦ The set of n-tuples from $\mathbb{R}$ is a vector space: $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$. The field is the set of real numbers $\mathbb{R}$.

♦ A set that is very important is the set of time limited functions that have a finite energy. The field is the set of complex numbers $\mathbb{C}$. We denote this set $L_2(T)$. The vector space is the set $L_2(T)$ over the set of complex numbers $\mathbb{C}$.

$x(t) = 0 \qquad t \notin [0,T]$ such that $\int_0^T |x(t)|^2\, dt < \infty$. When we consider it as a time function, we denote it $x(t)$ and when we consider it as a vector we designate it $\mathbf{x}$.

1.2 Linear independence

When we work with vectors, it is convenient to be able to express vectors as linear combination of other vectors. A vector $\mathbf{y}$ is a linear combination of the following n vectors $\mathbf{x}_1$, $\mathbf{x}_2$, …, $\mathbf{x}_n$ if we can express it as:

$\mathbf{y} = \sum_{k=1}^{n} \alpha_k \mathbf{x}_k$ where the scalars $\alpha_1, \alpha_2, \ldots, \alpha_n$ are not all zero.

Now, a set $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n\}$ of vectors from V forms a linearly independent set if no one of the vectors can be expressed as a linear combination of the others. This means that $\sum_{k=1}^{n} \alpha_k \mathbf{x}_k = 0$ if and only if $\alpha_1 = \alpha_2 = \ldots = \alpha_n = 0$.

Dimension of the space

In a vector space, the maximum number of linearly independent vectors is called the dimension of the space. Some spaces are finite dimensional. The vectors that represent fields in electromagnetic theory are three dimensional vectors. However, function spaces are usually infinite dimensional spaces.

Example:

Consider the set of complex functions of the real variable t defined as follows:

$$\varphi_k(t) = \begin{cases} \exp(jkt) & 0 \le t \le 2\pi \\ 0 & \text{elsewhere} \end{cases}$$

We can show that $\sum_{k=-\infty}^{+\infty} \alpha_k \varphi_k(t) = 0$ if and only if $\alpha_k = 0 \quad \text{for all } k \in \mathbb{Z}$.

Basis (plur. Bases)

When we possess a set $B = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ of vectors from V where $n = \dim V$, we can express any vector from V as a linear combination of the vectors composing B. We say that the set of vectors B forms a "basis". So, for $\forall \mathbf{x} \in V$, we can write $\mathbf{x} = \sum_{k=1}^{n} \alpha_k \mathbf{u}_k$. We say that the set B spans the space V and we can write also $V = \text{span}\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$. At that time, the vectors can be represented by the sequence of coefficients $\mathbf{x} = (\alpha_1, \alpha_2, \ldots, \alpha_n)$.

Example:

Consider the infinite set of functions defined previously. Any function that is nonzero only in the interval $[0, 2\pi]$ and that satisfies the Dirichlet conditions (see EE311 course) can be represented as the following linear combination:

$$x(t)=\sum_{k=-\infty}^{+\infty}\alpha_k \exp\left(jkt\right) \tag{1}$$

Equation (1) is nothing but the Fourier series representation of the signal $x(t)$. This signal is represented using the above basis and its coordinates are the Fourier coefficients α_k.

$$\alpha_k=\frac{1}{2\pi}\int_0^{2\pi} x(t)\exp\left(-jkt\right)dt \tag{2}$$

The vector **x** can then be represented by the infinite sequence:

$$\mathbf{x}=\left(\cdots,\alpha_{-1},\alpha_0,\alpha_1,\cdots,\alpha_k,\cdots\right)$$

Subspace

If we are given a vector space *V* over the field *F*, we can define a subspace as follows:

W is a subspace of *V* if it is a subset of *V* closed under the addition of vectors and multiplication by scalars from *F*. In finite spaces, we always have $\dim W < \dim V$.

1.3 Metric space and norm

<u>Distance</u>

If we want to compare elements from a given space, we have to define the notion of metric or distance.

The distance between two elements *a* and *b* of a set *S* is the positive real number $d(a, b)$ satisfying:

1. $\forall a,b\in S \quad d(a,b)\geq 0$, with $d(a,b)=0$ if and only if $a=b$
2. $\forall a,b,c\in S \quad d(a,c)\leq d(a,b)+d(b,c)$ (triangular inequality)

The set *S* is called a metric space.

<u>Norm</u>

If now the set *S* possesses the structure of a vector space over $\mathbb{C}$, we can define the length of the vectors using the concept of norm. The norm of a vector **x** is the positive real number $\|\mathbf{x}\|$ satisfying:

1. $\forall \mathbf{x}\in V \quad \|\mathbf{x}\|\geq 0$, with $\|\mathbf{x}\|=0$ if and if $\mathbf{x}=0$
2. $\forall \mathbf{x},\mathbf{y}\in V \quad \|\mathbf{x}+\mathbf{y}\|\leq\|\mathbf{x}\|+\|\mathbf{y}\|$
3. $\forall \mathbf{x}\in V$ *and* $\forall\alpha\in\mathbb{C} \quad \|\alpha\mathbf{x}\|=|\alpha|\|\mathbf{x}\|$

When a norm is defined in a vector space, we can deduce from it a distance between points in the space. If we consider that a vector is constituted by 2 points: Its origin (at the origin of the basis) and its tip, we can use it to represent points in the space spanned by the basis. So, the vector **x** represents also the point in the above space having as coordinates the coordinates of **x**. So, given two points represented by **x** and **y**, the distance between x and y can be expressed by:

$$d(\mathbf{x},\mathbf{y}) = \|\mathbf{x}-\mathbf{y}\| \tag{3}$$

Now if a metric space has the structure of a vector space over $\mathbb{C}$, a norm can be defined using the distance. Since a vector x is represented by two points (the origin and its tip) and the origin is represented by zero, then we can write:

$$\|\mathbf{x}\| = d(0,\mathbf{x}) \tag{4}$$

1.4 Inner product

The concept of inner product can be defined only for vectors spaces defined over the field of complex (real) numbers $\mathbb{C}$ ($\mathbb{R}$). Given a vectors space V defined over $\mathbb{C}$, the inner product of two vectors is defined as follows:

Let $\mathbf{x},\mathbf{y} \in V$, the inner product of **x** and **y** is the complex number $(\mathbf{x},\mathbf{y})$ satisfying:

1. $\forall \mathbf{x} \in V \quad (\mathbf{x},\mathbf{x}) \geq 0$, with $(\mathbf{x},\mathbf{x}) = 0$ if and only if **x** = 0
2. $\forall \mathbf{x},\mathbf{y} \in V \quad (\mathbf{x},\mathbf{y}) = (\mathbf{y},\mathbf{x})^*$
3. $\forall \mathbf{x},\mathbf{y},\mathbf{z} \in V \;\; and \;\; \alpha,\beta \in \mathbb{C} \quad (\alpha\mathbf{x}+\beta\mathbf{y},\mathbf{z}) = \alpha(\mathbf{x},\mathbf{z}) + \beta(\mathbf{y},\mathbf{z})$

Example:

Consider the two dimensional real vectors represented by their coordinates: $\vec{\mathbf{x}} = (x_1, x_2)$ where x_1, $x_2 \in \mathbb{R}$. The inner product of the two vectors $\vec{\mathbf{x}}$ and $\vec{\mathbf{y}}$ is usually written as $\vec{\mathbf{x}}.\vec{\mathbf{y}} = x_1y_1 + x_2y_2 = |\vec{\mathbf{x}}||\vec{\mathbf{y}}|\cos(\vec{\mathbf{x}},\vec{\mathbf{y}})$. It is also called "dot product".

If we consider the space $L_2(T)$ defined previously, the inner product is defined as:

$$(\mathbf{x},\mathbf{y}) = \int_0^T x(t)y^*(t)dt \tag{5}$$

<u>**Euclidian Norm**</u>

The properties of the inner product can be used to define a norm in the vector space: the Euclidian norm

$$\|\mathbf{x}\| = \sqrt{(\mathbf{x},\mathbf{x})} \tag{6}$$

In the two dimensional space shown above, the length of a vector $\overrightarrow{\mathbf{x}}$ having as coordinates x_1 and x_2 is $\|\vec{\mathbf{x}}\| = \sqrt{x_1^2 + x_2^2}$

In the space $L_2(T)$, the norm of a function is:

$\|\mathbf{x}\|^2 = \int_0^T |x(t)|^2 \, dt$ and this is the energy of the signal $x(t)$.

Properties of the inner product

Cauchy-Schwartz inequality

The Cauchy-Schwartz inequality is an important property of the inner product. It is commonly used to solve some optimization problems that occur in communication theory. It is also used to define the concept of an angle between two vectors.

Given two non-zero vectors **x** and **y** from the vector space V defined over $\mathbb{C}$, the following inequality applies:

$|(\mathbf{x},\mathbf{y})| \le \|\mathbf{x}\|\|\mathbf{y}\|$ with equality if $\mathbf{x} = \alpha\mathbf{y}$, α being a non-zero scalar.

Proof:

Let **x** and **y** be two nonzero vectors from the vector space V defined over $\mathbb{C}$ and the arbitrary nonzero scalar λ, consider the positive number:

$$D(\lambda) = \|\mathbf{x} - \lambda\mathbf{y}\|^2 = (\mathbf{x} - \lambda\mathbf{y}, \mathbf{x} - \lambda\mathbf{y})$$

Developing the product, we obtain:

$$\begin{aligned} D(\lambda) &= \|\mathbf{x}\|^2 - \lambda(\mathbf{y},\mathbf{x}) - \lambda^*(\mathbf{x},\mathbf{y}) + |\lambda|^2\|\mathbf{y}\|^2 \\ &= \|\mathbf{x}\|^2 - \lambda(\mathbf{x},\mathbf{y})^* - \lambda^*(\mathbf{x},\mathbf{y}) + |\lambda|^2\|\mathbf{y}\|^2 \end{aligned}$$

λ being arbitrary, let $\lambda = \dfrac{(\mathbf{x},\mathbf{y})}{\|\mathbf{y}\|^2}$. Using this value, $D(\lambda)$ becomes:

$D(\lambda) = \|\mathbf{x}\|^2 - \dfrac{|(\mathbf{x},\mathbf{y})|^2}{\|\mathbf{y}\|^2}$. Since $D(\lambda) \ge 0$, the inequality is proved.

We will have equality if $D(\lambda) = 0$. In this case, $\mathbf{x} = \lambda\mathbf{y}$.

♦

Using the Cauchy-Schwartz inequality, we can define the angle between the vector **x** and **y** as:

$$\cos\theta = \frac{(\mathbf{x},\mathbf{y})}{\|\mathbf{x}\|\|\mathbf{y}\|} \qquad (7)$$

If $\mathbf{x} = \lambda\mathbf{y}$, the angle between the two vectors will be 0 or 180° ($\cos\theta = \pm 1$). This means that the two vectors are collinear. If the two vectors are nonzero, they will be orthogonal if (**x**, **y**) = 0 ($\theta = \pm\frac{\pi}{2}$). We use the notation $\mathbf{x} \perp \mathbf{y}$ to say **x** orthogonal with **y**.

In the $L_2(T)$ space, the Cauchy-Schwartz inequality can be expressed as:

$$\left|\int_0^T x(t)y^*(t)dt\right|^2 \leq \int_0^T |x(t)|^2 dt \int_0^T |y(t)|^2 dt$$

Pythagorean Theorem

If two vectors **x** and **y** are orthogonal, they satisfy the Pythagorean theorem:

$$\|\mathbf{x}+\mathbf{y}\|^2 = \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2$$

Proof:

$\|\mathbf{x}+\mathbf{y}\|^2 = (\mathbf{x}+\mathbf{y}, \mathbf{x}+\mathbf{y}) = \|\mathbf{x}\|^2 + (\mathbf{x},\mathbf{y}) + (\mathbf{y},\mathbf{x}) + \|\mathbf{y}\|^2$. However, $\mathbf{x} \perp \mathbf{y}$, so (**x**, **y**) = (**y**, **x**) = 0.

♦

1.5 Orthonormal basis

In signal spaces (and even in general vector spaces) it is convenient to represent vectors using an "orthonormal basis". Consider a space V such that dim$V$ = n. (n can be infinite). The set of vectors $\{\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_2, \cdots, \boldsymbol{\varphi}_n\}$ forms an orthonormal basis if the vectors satisfy:

$$(\boldsymbol{\varphi}_k, \boldsymbol{\varphi}_i) = \begin{cases} 0 & k \neq i \\ 1 & k = i \end{cases}$$

These vectors are linearly independent and their number is equal to the dimension of the space. So, we can write $V = \text{span}\{\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_2, \cdots, \boldsymbol{\varphi}_n\}$. This means that any vector in V can be expressed as:

$$\mathbf{x} = \sum_{k=1}^{n} \alpha_k \boldsymbol{\varphi}_k \tag{8}$$

One advantage of an orthonormal basis is the simple expression of the coefficients. If we compute the inner product of **x** with one basis vector, we obtain:

$$(\mathbf{x}, \boldsymbol{\varphi}_m) = \left(\sum_{k=1}^{n} \alpha_k \boldsymbol{\varphi}_k, \boldsymbol{\varphi}_m\right) = \sum_{k=1}^{n} \alpha_k (\boldsymbol{\varphi}_k, \boldsymbol{\varphi}_m)$$

So:

$$\alpha_m = (\mathbf{x}, \boldsymbol{\varphi}_m) \tag{9}$$

In the case of the $L_2(T)$ space, the expression of the coefficients is then

$$\alpha_m = \int_0^T x(t)\varphi_m(t)dt \tag{10}$$

In the signal space $L_2(T)$, the expression (8) is called Fourier decomposition of the signal $x(t)$ and is written as:

$$x(t) = \sum_{k=1}^{n} \alpha_k \varphi_k(t) \tag{11}$$

The coefficients α_k are called the Fourier coefficients.

Orthogonal spaces

Consider a vector space V of dimension n. It can be decomposed into the "sum" of two "orthogonal" subspaces V_1 and V_2 such that:

$$\forall \mathbf{x}_1 \in V_1 \quad and \quad \forall \mathbf{x}_2 \in V_2 \qquad \mathbf{x}_1 \perp \mathbf{x}_2$$

$$\forall \mathbf{x} \in V \quad \exists \mathbf{x}_1 \in V_1 \quad \exists \mathbf{x}_2 \in V_2 \quad | \, \mathbf{x} = \mathbf{x}_1 + \mathbf{x}_2$$

Using orthonormal bases, we can easily decompose any vector into the two above components. Assume that we are given an orthonormal basis such that $V = span\{\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_2, \cdots, \boldsymbol{\varphi}_n\}$ and let us define two subspaces $V_1 = span\{\boldsymbol{\varphi}_1, \cdots, \boldsymbol{\varphi}_m\}$ and $V_2 = span\{\boldsymbol{\varphi}_{m+1}, \cdots, \boldsymbol{\varphi}_n\}$. It is clear that V_1 and V_2 are orthogonal. If we consider any vector of V, its first m coordinates define an element of V_1 while the last $n - m$ coordinates define an element of V_2. So:

$\forall \mathbf{x} \in V \quad \mathbf{x}_1 = \sum_{k=1}^{m} \alpha_k \boldsymbol{\varphi}_k \in V_1 \quad and \quad \mathbf{x}_2 = \sum_{k=m+1}^{n} \alpha_k \boldsymbol{\varphi}_k \in V_2$. The coefficients α_k are computed using equation(9): $\alpha_k = (\mathbf{x}, \boldsymbol{\varphi}_k)$. The vector $\mathbf{x}_1$ is the orthogonal projection of $\mathbf{x}$ on V_1 while the vector $\mathbf{x}_2$ is the orthogonal projection of $\mathbf{x}$ on V_2.

The Gram-Schmidt procedure

The Gram-Schmidt procedure is a procedure for deriving an orthonormal basis from a set of M vectors spanning a space of dimension $K \le M$. So, we are given the set $\{\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_M\}$ of vectors, not necessarily linearly independent.

The procedure follows the subsequent steps:

1. $\boldsymbol{\varphi}_1 = \frac{\mathbf{x}_1}{\|\mathbf{x}_1\|}$; this is an initialization step.
2. For k = 2 to M let:

   $\mathbf{g}_k = \mathbf{x}_k - \sum_{i=1}^{k-1} (\mathbf{x}_k, \boldsymbol{\varphi}_i) \boldsymbol{\varphi}_i$; We subtract from $\mathbf{x}_k$ its projection on the $k-1$ dimensional space spanned by the already found basis vectors. This implies that $\mathbf{g}_k$ belongs to a space that is orthogonal to the space spanned by the $k - 1$ basis vectors $\boldsymbol{\varphi}_k$. If $\mathbf{g}_k$ is zero, the vector $\mathbf{x}_k$ must be discarded. It belongs to the previous space and it is not linearly independent. The basis vector is then: $\boldsymbol{\varphi}_k = \frac{\mathbf{g}_k}{\|\mathbf{g}_k\|}$.

At the end of step 2, we will have a set of K orthonormal basis vectors. If the set $\{\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_M\}$ is composed of linearly independent vectors, we will have $K = M$. Otherwise, the dimension of the space will be smaller than the number of given vectors.

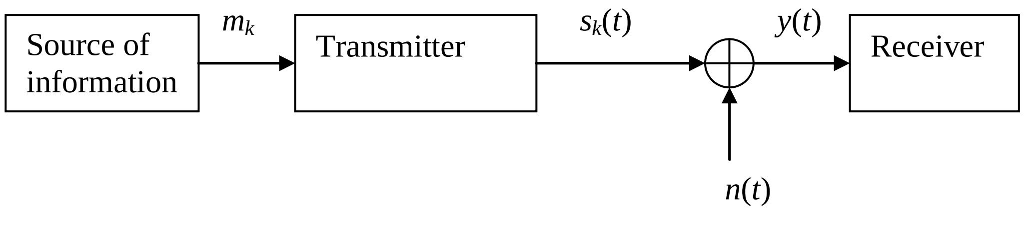
Expression of the inner product in an orthonormal space

Let us consider two vectors **x** and **y** with coordinates (x_1, x_2, …, x_K) and (y_1, y_2, …, y_K) in an orthonormal basis. It is easy to show that:

$$(\mathbf{x}, \mathbf{y}) = \left(\sum_{k=1}^{K} x_k \boldsymbol{\varphi}_k, \sum_{j=1}^{K} y_j \boldsymbol{\varphi}_j \right) = \sum_{k=1}^{K} x_k y_k^* \qquad (12)$$

1.6 Waveform communication system

Let us consider the following communication system



We are given a source of information that can produce M different symbols. The transmitter assigns a signal $s_k(t)$ to the message m_k. The signals belong to the space $L_2(T)$. This means that the duration of a symbol is T seconds and the rate of symbols (it is called the "baud rate") is $r = \frac{1}{T}$. The total number of signals produced by the transmitter is M. These signals belong to a finite

dimensional space. In fact, we can use these M signals to derive an orthonormal basis using the Gram-Schmidt procedure. The derived basis is composed of $K \leq M$ orthonormal signals $\varphi_k(t)$. Let us call this space S_K, $S_K = span\{\boldsymbol{\varphi}_1, \cdots, \boldsymbol{\varphi}_K\}$.
The signals generated by the transmitter can be expressed in this space as:

$$s_i(t) = \sum_{k=1}^{K} \alpha_{i,k} \varphi_k(t) \qquad ; \qquad i = 1, \ldots, M$$

So, we have a correspondence between the signal $s_i(t)$ and the vector $\mathbf{s}_i = \left(\alpha_{i,1}, \alpha_{i,2}, \cdots, \alpha_{i,K}\right)$.

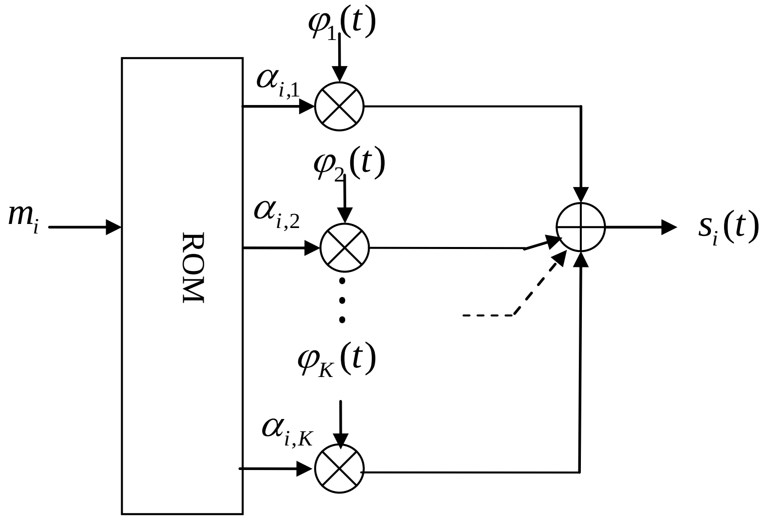


Fig.3- 1 Possible transmitter structure

Fig.3- 1 shows a possible implementation of a transmitter. The symbol m_i selects a set of K coefficients. The waveform produced by the transmitter is a sequence of finite time signals, each one being a weighted sum of basis functions.

The noise signal $n(t)$ is assumed to be a white Gaussian noise with a power spectrum $S_n(f) = \frac{N_0}{2}$. The received signal $y(t)$ is the sum of the transmitted signal and this noise signal. Despite the fact that the signal at the output of the transmitter belongs to a finite dimensional space, the signal at the input of the receiver is a signal which belongs to an infinite dimensional space. We can see this by developing the noise process.

Consider the following orthonormal basis: $\{\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_2, \ldots \boldsymbol{\varphi}_K, \boldsymbol{\varphi}_{K+1}, \ldots\}$ where the first K basis vectors are the vector representation of the basis functions used to represent the signals generated by the transmitter. We complete this basis by an arbitrary set of orthonormal functions to span an infinite dimensional space. In this basis, we can show that the noise process is expressed as:

$$n(t) = \sum_{k=1}^{\infty} N_k \varphi_k(t) = \sum_{k=1}^{K} N_k \varphi_k(t) + \sum_{k=K+1}^{\infty} N_k \varphi_k(t) \tag{13}$$

Equation (13) shows that the noise process is the sum of two orthogonal processes. The first one belongs to the same space as the signals generated by the transmitter (S_K). The second one is orthogonal to all these signals. The coordinates N_k are random variables and they are the projections of the noise vector on the basis functions.

$$N_k = (\mathbf{n}, \boldsymbol{\varphi}_k) = \int_0^T n(t)\varphi_k(t)dt \tag{14}$$

Since the noise process is Gaussian, these variables are Gaussian random variables. We can derive their statistics as follows:

$$E[N_k] = E\left[\int_0^T n(t)\varphi_k(t)dt\right] = \int_0^T E[n(t)]\varphi_k(t)dt = 0$$

So, they are zero mean. Their covariance is then:

$$E\left[N_k N_j\right] = E\left[\int_0^T n(t)\varphi_k(t)dt \int_0^T n(u)\varphi_j(u)du\right] = \int_0^T \int_0^T E[n(t)n(u)]\varphi_k(t)\varphi_j(u)dtdu$$

The noise process is white. Its autocorrelation function is:

$$R_n(t-u) = E[n(t)n(u)] = \frac{N_0}{2}\delta(t-u)$$

The covariance is then:

$$E\left[N_k N_j\right] = \int_0^T \frac{N_0}{2}\varphi_k(t)\varphi_j(t)dt$$

Since the basis functions are orthonormal, the above covariance is:

$$E\left[N_k N_j\right] = \begin{cases} \frac{N_0}{2} & k = j \\ 0 & k \neq j \end{cases}$$

So, the components of the noise process are independent Gaussian random variables with zero mean and variances equal to $\frac{N_0}{2}$. These components will be the same in any orthonormal basis. A white Gaussian noise is always represented by an infinite dimensional vector with coordinates that are independent zero mean Gaussian vector with variance equal to $\frac{N_0}{2}$.

The signal observed by the receiver is the signal $y(t)$. It consists of the sum of a finite dimensional vector ($s_i(t)$) and the infinite dimensional vector representing the white noise process. However, we can express this received signal as a sum of two orthogonal signals.

$$y(t) = s_i(t) + n_1(t) + n_2(t) = z(t) + n_2(t)$$

where $n_1(t) = \sum_{k=1}^{K} N_k \varphi_k(t) \in S_K$ and $n_2(t) = \sum_{k=K+1}^{\infty} N_k \varphi_k(t) \perp S_K$. The signal $z(t)$ is $z(t) = s_i(t) + n_1(t)$. It is clear that $z(t) \in S_K$ and that $z(t) \perp n_2(t)$. We can show that the observation of the vector representing the signal $n_2(t)$ is not needed if we want to detect which message has been transmitted. So, at the receiver, we are going to observe only the signal $z(t)$ and not the signal $y(t)$. The signal $z(t)$ is the orthogonal projection of $y(t)$ on the space S_K and it can be represented by a finite dimensional vector $\mathbf{z} = (z_1, z_2, \ldots, z_K)$. The signal $y(t)$ on the other hand must be represented by an infinite dimensional vector $\mathbf{y} = (y_1, y_2, \ldots, y_K, y_{K+1}, \ldots)$ and the first K coordinates correspond to the vector $\mathbf{z}$.

We can now write the expression of the conditional pdf of the vector $\mathbf{z}$ given that the symbol m_i is transmitted. The coordinates of $\mathbf{z}$ are:

$z_k = \alpha_{i,k} + N_k$ for $k = 1, \ldots, K$. The numbers $\alpha_{i,k}$ are the coordinates of the signal $s_i(t)$. So, the coordinates z_k are independent Gaussian random variables with mean $\alpha_{i,k}$ and variance $\frac{N_0}{2}$. The conditional pdf of $\mathbf{z}$ is:

$$f_{\mathbf{z}|m_i}(\mathbf{z} \mid m_i) = \prod_{k=1}^{K} \frac{1}{\sqrt{2\pi}\sqrt{\frac{N_0}{2}}} \exp\left[-\frac{\left(z_k - \alpha_{i,k}\right)^2}{N_0} \right]$$

or:

$$f_{z|m_i}(z \mid m_i) = \frac{1}{\left(\pi N_0\right)^{\frac{K}{2}}} \exp\left[-\frac{1}{N_0} \sum_{k=1}^{K} \left(z_k - \alpha_{i,k}\right)^2 \right] \qquad (15)$$

We can use equation (15) to determine the structure of an MAP receiver. We have seen that the MAP rule is:

"Decide m_i if $P\left(m_i\right) f_{\mathbf{z}|m_i}(\mathbf{z} \mid m_i) > P\left(m_j\right) f_{\mathbf{z}|m_j}(\mathbf{z} \mid m_j)$ for all $j \neq i$; $j = 1, \ldots, M$"

Replacing and taking logarithms, we obtain:

$$\text{"Decide } m_i \text{ if } \left\{ \sum_{k=1}^{K} \left(z_k - \alpha_{i,k}\right)^2 - N_0 \ln P\left(m_i\right) \right\} \quad \min \text{"} \qquad (16)$$

The first term in the above expression is just the square of the distance between the vector $\mathbf{z}$ and the vector $\mathbf{s}_i$. So we can re-express the rule (16) as:

$$\left\{\|\mathbf{z}-\mathbf{s}_i\|^2 - N_0 \ln P(m_i)\right\} \quad \text{min} \tag{17}$$

If the symbols are equiprobable (ML receiver), the second term of (17) becomes irrelevant and the decision rule simplifies to:

$$\text{Decide } m_i \text{ such that } \|\mathbf{z}-\mathbf{s}_i\| \text{ min} \tag{18}$$

The receiver is called a "minimum distance classifier". In order to build the receiver, we must determine the coordinates of the vector **z**. We have seen that they are the first K coordinates of the vector **y**. From (9), the coordinates of **y** are given by:

$$y_k = z_k = (\mathbf{y}, \boldsymbol{\varphi}_k) = \int_0^T y(t)\varphi_k(t)dt \text{ for } k = 1, \ldots, K.$$

So, the receiver is composed of three cascaded sections. The first one computes the coordinates of the vector **z**. The second one is a distance computer and the third one applies a bias to take into account the unequal a priori probabilities.

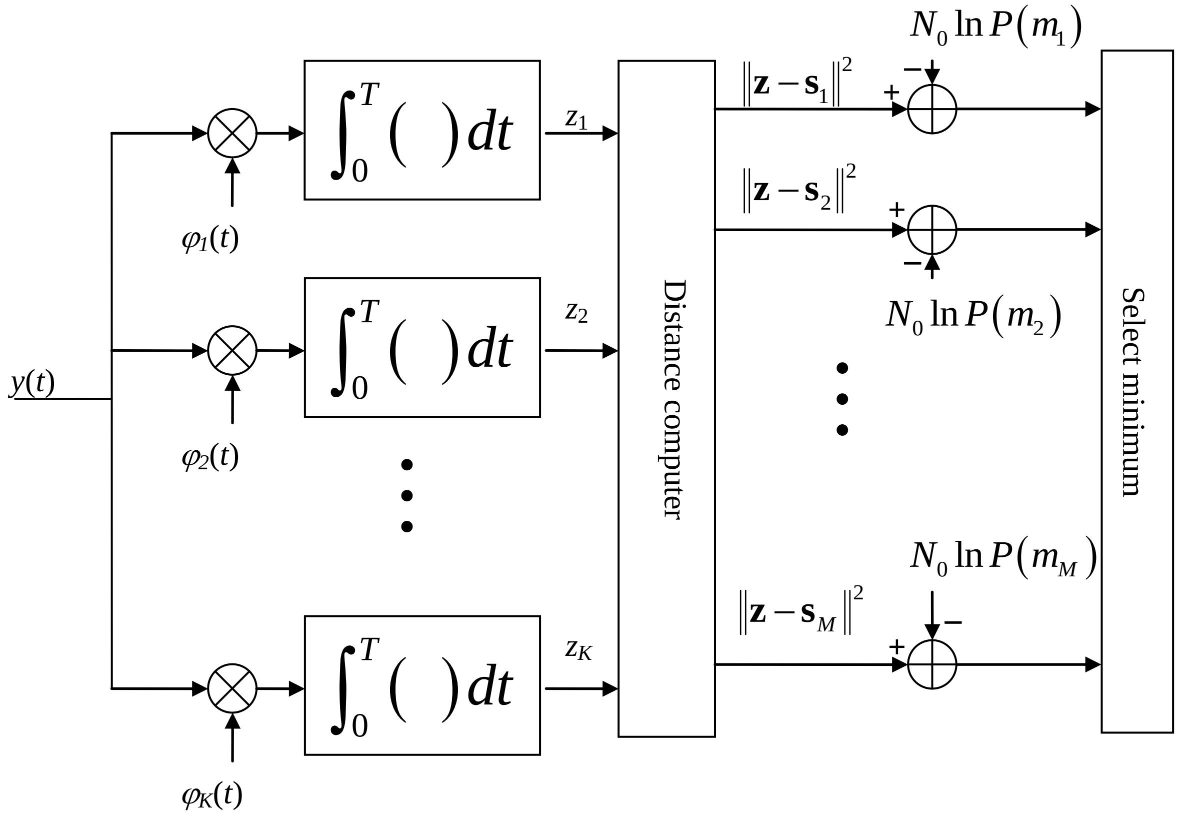


Fig.3- 2 Receiver structure using distance computer

The above structure uses K "correlators" that transform the random process $y(t)$ to the random vector **z**. The distance computer has K inputs (coordinates of **z**) and computes M distances (between the received vector **z** and the "prototypes" $\mathbf{s}_i$). If the symbols are equiprobable (this is often the case), the bias $-N_0 \ln P(m_i)$ is not required.

A different structure can be obtained if we develop the square of the distance.

$$\|\mathbf{z}-\mathbf{s}_i\|^2 = (\mathbf{z}-\mathbf{s}_i, \mathbf{z}-\mathbf{s}_i) = (\mathbf{z},\mathbf{z}) + (\mathbf{s}_i,\mathbf{s}_i) - 2(\mathbf{z},\mathbf{s}_i) \quad (19)$$

In the above expression, the squared norm of **z** does not depend on the message. The same value $\|\mathbf{z}\|^2$ will appear for all symbols. It can be eliminated. The inner product of $\mathbf{s}_i$ with itself is the energy of the signal $s_i(t)$.

$$(\mathbf{s}_i,\mathbf{s}_i) = \|\mathbf{s}_i\|^2 = \int_0^T s_i^2(t)dt = E_i$$

So, the expression (17) simplifies to:

$$\|\mathbf{z}-\mathbf{s}_i\|^2 - N_0 \ln P(m_i) = cste + E_i - 2(\mathbf{z},\mathbf{s}_i) - N_0 \ln P(m_i)$$

The decision rule becomes:

Decide m_i if $\left\{(\mathbf{z},\mathbf{s}_i) - \frac{E_i}{2} + \frac{N_0}{2}\ln P(m_i)\right\}$ max .

The inner product $(\mathbf{z},\mathbf{s}_i)$ can be computed using the signal $y(t)$ present at the input of the receiver because $y(t) = z(t) + n_2(t)$ and $n_2(t) \perp z(t)$. So:

$$(\mathbf{z},\mathbf{s}_i) = (\mathbf{y}-\mathbf{n}_2,\mathbf{s}_i) = (\mathbf{y},\mathbf{s}_i) = \int_0^T y(t)s_i(t)dt$$

Finally, the decision rule is:

$$\text{Decide } m_i \text{ if } \left\{\left[\int_0^T y(t)s_i(t)dt\right] - \frac{E_i}{2} + \frac{N_0}{2}\ln P(m_i)\right\} \text{ max} \quad (20)$$

The decision rule (20) can be implemented using the structure shown in Fig.3- 3. This structure uses M correlators and does not require a distance computer. If the symbols are equiprobable, the bias becomes only $\frac{E_i}{2}$. The correlation receiver has a simpler structure than the minimum distance one. However, it uses M correlators. In many cases, (PAM, MPSK and QAM) the dimension of the signal space is less or equal to two while the number of signals can be very high. In this case, the minimum distance receiver is better. However, when the dimensionality of the signal space is high and M is equal to K, it is better to use the correlation receiver structure.

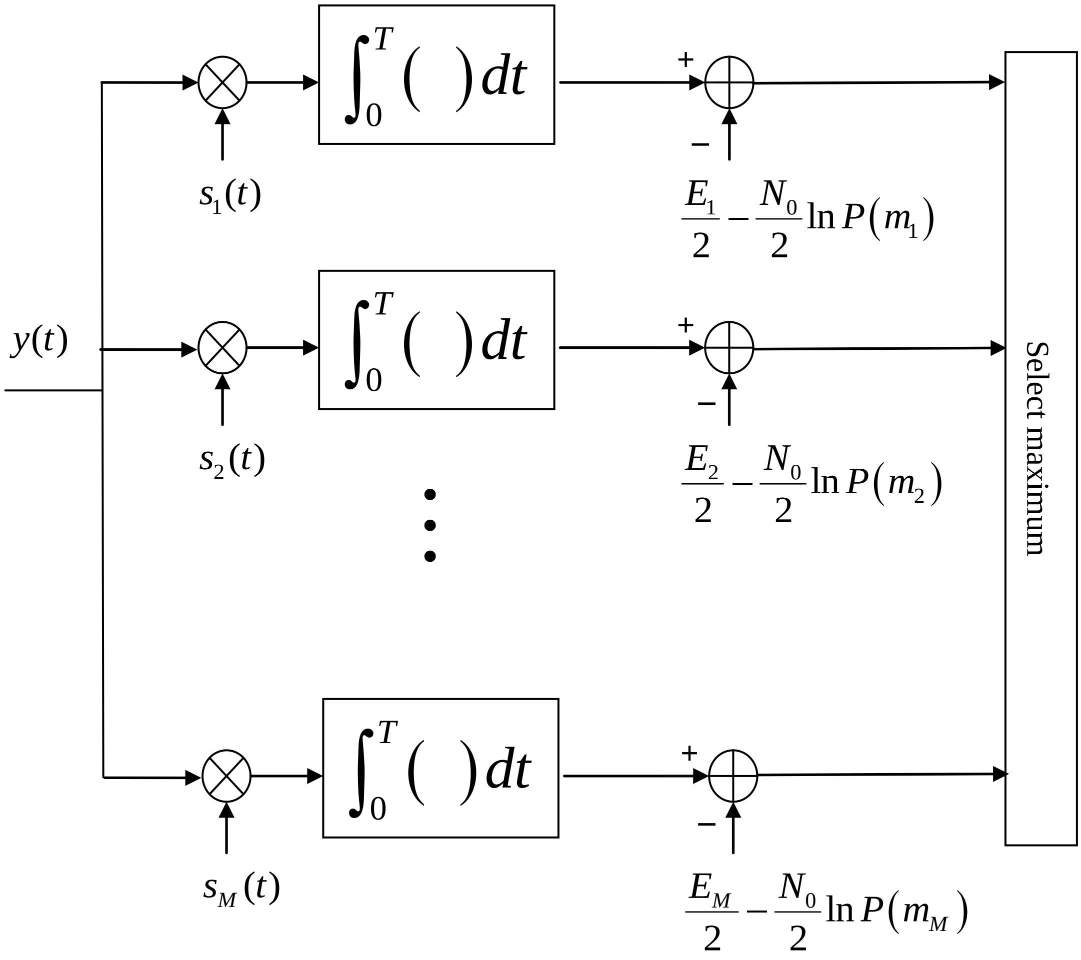


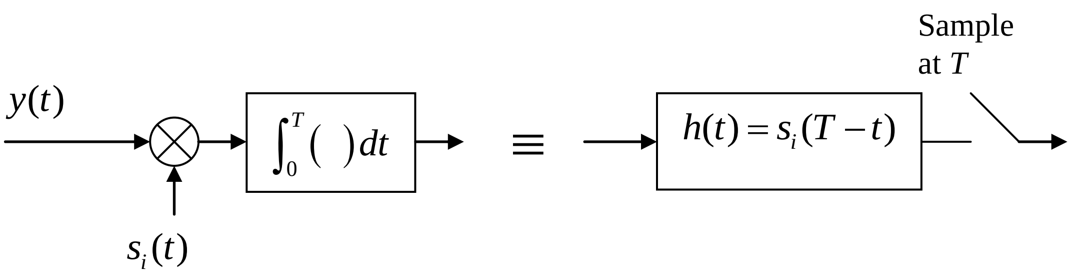
Fig.3- 3 Correlation receiver

The different correlators can be implemented by matched filters. We have already studied this equivalence in the matched filter course.

The correlator output is

$$\int_0^T y(t)s_i(t)dt = \int_0^T y(T-u)s_i(T-u)du = \left[\int_0^t s_i(T-u)y(t-u)du\right]_{t=T}$$

The last integral is the evaluation of the convolution of a filter with impulse response $h(t) = s_i(T-t)$ with the signal $y(t)$ at the time $t = T$. So, we obtain the following equivalence:



The decision rule divides the observation space into decision regions I_k such that $z \in L_k \Rightarrow$ decide m_k. If the symbols are equiprobable, the decision I_k is the set of points that are closer to the point $\mathbf{s}_k$ than to the other points.

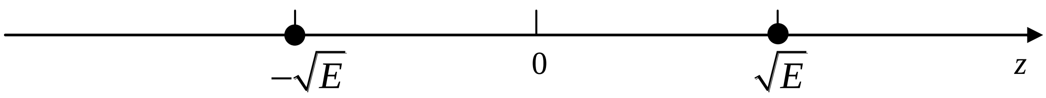
1.7 Some examples

For all examples, we assume equiprobable symbols.

Antipodal signaling

This communication system is a one-dimensional binary system. We use two signals:

$s_1(t) = \sqrt{E}\varphi(t)$ and $s_2(t) = -\sqrt{E}\varphi(t)$. The function $\varphi(t)$ has an energy equal to one. Using φ as a basis, the signal space is represented below.



We can use a minimum distance classifier. It uses only one correlator. Let the output of the correlator be called z. In this particular case, we don't have to compute the distance between the observed value z and the points $-\sqrt{E}$ and $\sqrt{E}$. The decision regions are separated by the origin. The optimum receiver is:

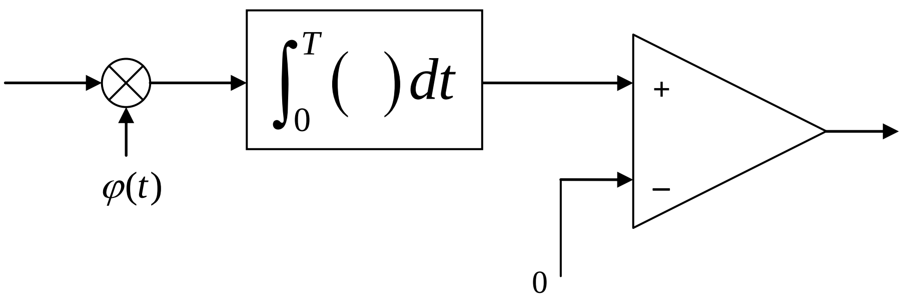


Fig.3- 4 Antipodal signalling receiver

Given that the distance between the mean of each distribution and the threshold is $A = \sqrt{E}$ and the variance is $\frac{N_0}{2}$, the probability of error is:

$$P(E) = \frac{1}{2}\mathrm{erfc}\left(\frac{A}{\sqrt{2}\sigma}\right) = \frac{1}{2}\mathrm{erfc}\sqrt{\frac{E}{N_0}} \qquad (21)$$

This is the same result as the one deduced using the matched filter derived in the previous chapter.

Pulse amplitude modulation (PAM)

The PAM system is also a one dimensional system. However, in this system, we use M different amplitudes. The amplitudes are equally spaced and are such that the average value is zero. Furthermore, we assume that $M = 2^N$ in order to be able to encode the different levels in binary. So, M levels correspond to $N = \log_2 M$ binary digits. The level separation is A.

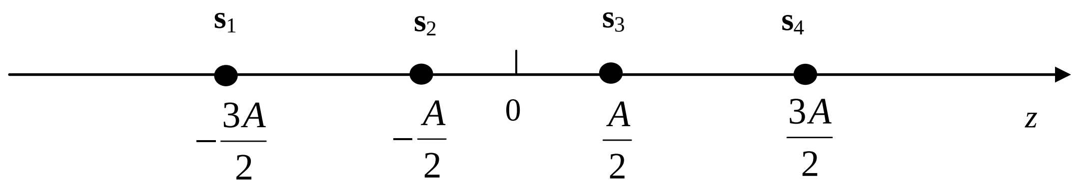


Fig.3- 5 PAM with M = 4

The decision regions are intervals separated by thresholds occurring midway between the signal levels. For example, the region I_2 corresponding to the signal $\mathbf{s}_2$ is the interval $[-A, 0]$. The region I_1 corresponding to $\mathbf{s}_1$ is the interval $]-\infty, -A]$. For the general case of M different signal levels, we are going to have M – 1 thresholds.

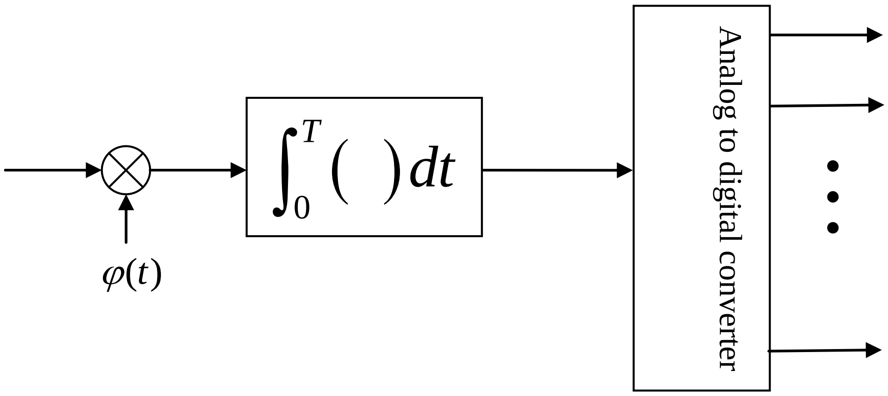


Fig.3- 6 PAM receiver

The above figure shows a typical PAM receiver. The analog to digital converter encodes directly the signal levels into binary. It can be implemented using $M - 1$ comparators followed by a priority encoder or it can be implemented using a successive approximation analog to digital converter. The different levels are usually encoded in Gray code, so that an error in one level will cause only one bit to be in error.

To compute the probability of error, we have to take into consideration whether the level is an intermediate level (between two thresholds) or an extreme one (bounded by one threshold only). We have two extreme levels: $\mathbf{s}_1$ and $\mathbf{s}_M$. The conditional probability of error is the area of a tail of a Gaussian with a threshold away from the mean by $\frac{A}{2}$ and a variance $\frac{N_0}{2}$. So:

$$P(E \mid m_1) = P(E \mid m_M) = \frac{1}{2}\operatorname{erfc}\left(\frac{A}{2\sqrt{2}\sqrt{\frac{N_0}{2}}}\right) = \frac{1}{2}\operatorname{erfc}\left(\frac{A}{2\sqrt{N_0}}\right)$$

For the intermediate levels, we have two thresholds, so for the levels $\mathbf{s}_2$ up to $\mathbf{s}_{M-1}$, the conditional probability of error is the probability of being away from the mean by $\frac{A}{2}$ on both sides. So:

$$P(E \mid m_2) = \ldots = P(E \mid m_{M-1}) = \operatorname{erfc}\left(\frac{A}{2\sqrt{2}\sqrt{\frac{N_0}{2}}}\right) = \operatorname{erfc}\left(\frac{A}{2\sqrt{N_0}}\right)$$

Finally, the probability of error is the average:

$$P(E) = \sum_{k=1}^{M} P(m_k) P(E \mid m_k) = \left(\frac{M-1}{M}\right)\operatorname{erfc}\left(\frac{A}{2\sqrt{N_0}}\right) \qquad (22)$$

We can express the amplitude A as a function of the average energy of the signals:

$$E = \frac{1}{M}\sum_{k=1}^{M} \|\mathbf{s}_k\|^2 = \frac{1}{M}\sum_{k=1}^{M} |\alpha_k|^2$$

where the coefficients α_k are the amplitudes of the signals.

For the PAM signaling where the spacing between the points has a value of A, the different amplitudes are: $\pm\frac{A}{2}(2k-1)$ for k = 1 to $\frac{M}{2}$. The average energy is then:

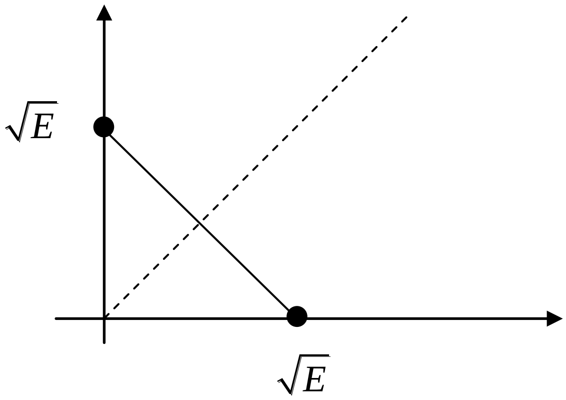
$$E = \frac{2}{M}\sum_{k=1}^{\frac{M}{2}} \frac{A^2}{4}(2k-1)^2 = \frac{M^2-1}{12}A^2$$

Replacing in (22), we obtain:

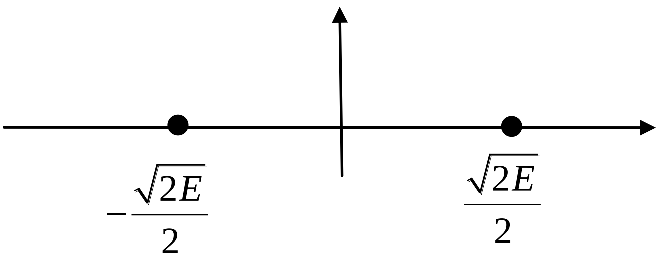
$$P(E)=\left(\frac{M-1}{M}\right)\text{erfc}\sqrt{\frac{3}{M^2-1}\frac{E}{N_0}} \tag{23}$$

Orthogonal signaling

This is a two dimensional signaling scheme. The two signals are: $s_1(t)=\sqrt{E}\varphi_1(t)$ and $s_2(t)=\sqrt{E}\varphi_2(t)$. The two basis functions are orthonormal.



The coordinates of the two vectors are $\left(\sqrt{E},0\right)$ and $\left(0,\sqrt{E}\right)$. We can show that the probability of error is independent on translations and rotations in the signal space. So, under proper translation and rotation, the above signal space is equivalent to the following one:



The above figure is the same as the antipodal case. The probability of error is:

$$P\left(E\right)=\frac{1}{2}\text{erfc}\left(\frac{A}{\sqrt{2}\sigma}\right)$$

where $A=\frac{\sqrt{2E}}{2}=\sqrt{\frac{E}{2}}$ and $\sigma^2=\frac{N_0}{2}$

The probability of error is then:

$$P(E)=\frac{1}{2}\operatorname{erfc}\sqrt{\frac{E}{2N_0}} \tag{24}$$

An example of orthogonal signaling is the orthogonal FSK system. The receiver can be implemented as follows:

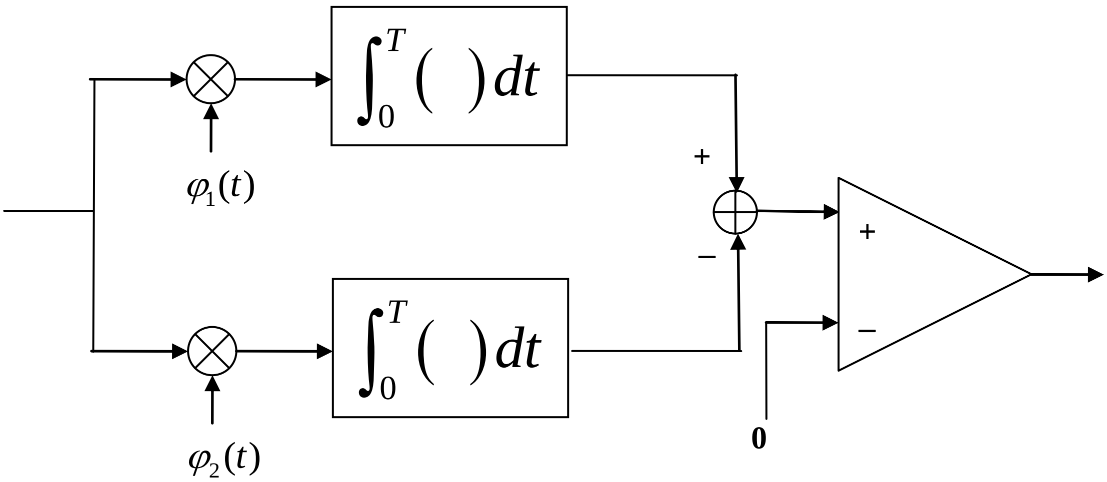


Fig.3- 7 orthogonal receiver structure

Non coherent FSK

In this communication system, the phase of the received waveform is unknown. It will be modeled as a uniformly distributed random variable over $[0,2\pi]$. We assume also that the frequencies are selected so that the two carriers are orthogonal. The receiver observes one of the following two signals:

$s_1(t)=A\cos(\omega_1 t+\Theta)$ and $s_2(t)=A\cos(\omega_2 t+\Theta)$

Θ is a random variable uniformly distributed over $[0, 2\pi]$. The above representation corresponds to an infinite number of signals. Consider the following four orthonormal functions:

$$\varphi_1(t)=\sqrt{\frac{2}{T}}\cos\omega_1 t,\ \varphi_2(t)=\sqrt{\frac{2}{T}}\sin\omega_1 t,\ \varphi_3(t)=\sqrt{\frac{2}{T}}\cos\omega_2 t,\ \varphi_4(t)=\sqrt{\frac{2}{T}}\sin\omega_2 t$$

and let θ be a realization of the random variable Θ. We can represent the above signals in the four dimensional space spanned by $\boldsymbol{\varphi}_1$, $\boldsymbol{\varphi}_2$, $\boldsymbol{\varphi}_3$, $\boldsymbol{\varphi}_4$. The corresponding vectors are:

$$\mathbf{s}_1=\left(A\sqrt{\frac{T}{2}}\cos\theta,-A\sqrt{\frac{T}{2}}\sin\theta,0,0\right)$$

$$\mathbf{s}_2 = \left(0, 0, A\sqrt{\frac{T}{2}}\cos\theta, -A\sqrt{\frac{T}{2}}\sin\theta\right)$$

The vector observed by the receiver is one of the above signals plus noise. We have seen that the components of noise are independent Gaussian random variables with zero mean and a variance equal to $\frac{N_0}{2}$. So, the four dimensional observed vector is the following Gaussian random vector: $\mathbf{z} = (z_1, z_2, z_3, z_4)$.

When ω_1 is transmitted and $\Theta = \theta$ is observed, $\mathbf{z}$ has the following pdf:

$$f_{\mathbf{z}|\omega_1,\theta}(\mathbf{z} \mid \omega_1, \theta) = \frac{1}{(\pi N_0)^2} \exp\left\{-\frac{1}{N_0}\left[\left(z_1 - A\sqrt{\frac{T}{2}}\cos\theta\right)^2 + \left(z_2 + A\sqrt{\frac{T}{2}}\sin\theta\right)^2 + z_3^2 + z_4^2\right]\right\}$$

Using $E = \frac{A^2 T}{2}$, the above expression becomes:

$$f_{\mathbf{z}|\omega_1,\theta}(\mathbf{z} \mid \omega_1, \theta) = \frac{1}{(\pi N_0)^2} \exp\left\{-\frac{1}{N_0}\left[\left(z_1 - \sqrt{E}\cos\theta\right)^2 + \left(z_2 + \sqrt{E}\sin\theta\right)^2 + z_3^2 + z_4^2\right]\right\}$$

This expression becomes:

$$f_{\mathbf{z}|\omega_1,\theta}(\mathbf{z} \mid \omega_1, \theta) = \frac{1}{(\pi N_0)^2} \exp\left\{-\frac{1}{N_0}\left[z_1^2 + z_2^2 + z_3^2 + z_4^2 + E - 2\sqrt{E}\left(z_1\cos\theta - z_2\sin\theta\right)\right]\right\}$$

When ω_2 is transmitted and $\Theta = \theta$ is observed, $\mathbf{z}$ has the following pdf:

$$f_{\mathbf{z}|\omega_2,\theta}(\mathbf{z} \mid \omega_2, \theta) = \frac{1}{(\pi N_0)^2} \exp\left\{-\frac{1}{N_0}\left[z_1^2 + z_2^2 + z_3^2 + z_4^2 + E - 2\sqrt{E}\left(z_3\cos\theta - z_4\sin\theta\right)\right]\right\}$$

A change of variable (from rectangular to polar) produces an expression of the pdf that is more informative.

So, let $z_1 = r_1\cos\psi_1$, $z_2 = r_1\sin\psi_1$, $z_3 = r_2\cos\psi_2$ and $z_4 = r_2\sin\psi_2$. The above pdf's become:

$$f\left(r_1, \psi_1, r_2, \psi_2 \mid \omega_1, \theta\right) = \frac{r_1 r_2}{(\pi N_0)^2} \exp\left\{-\frac{1}{N_0}\left[r_1^2 + r_2^2 + E - 2\sqrt{E} r_1 \cos\left(\theta + \psi_1\right)\right]\right\}$$

$$f\left(r_1, \psi_1, r_2, \psi_2 \mid \omega_2, \theta\right) = \frac{r_1 r_2}{(\pi N_0)^2} \exp\left\{-\frac{1}{N_0}\left[r_1^2 + r_2^2 + E - 2\sqrt{E} r_2 \cos\left(\theta + \psi_2\right)\right]\right\}$$

If we can measure the phase of the carrier, we can use the above expressions for an MAP receiver design. However, we don't have that information. The solution is to take a decision after averaging the above pdf's over θ.

$$f\left(r_1,\psi_1,r_2,\psi_2 \mid \omega_1\right)=\frac{1}{2\pi}\int_0^{2\pi} f\left(r_1,\psi_1,r_2,\psi_2 \mid \omega_1,\theta\right)d\theta$$

$$=\frac{r_1 r_2}{\left(\pi N_0\right)^2}\exp\left\{-\frac{1}{N_0}\left[r_1^2+r_2^2+E\right]\right\}\frac{1}{2\pi}\int_0^{T}\exp\left[\frac{2\sqrt{E}}{N_0}r_1\cos\left(\theta+\psi_1\right)\right]d\theta$$

We use the fact that:

$\frac{1}{2\pi}\int_0^{2\pi}\exp\left(z\cos\lambda\right)d\lambda=I_0(z)$, I_0 is the modified Bessel function of the first kind of order zero. The pdf is:

$$f(r_1,\psi_1,r_2,\psi_2 \mid \omega_1)=\frac{r_1 r_2}{\left(\pi N_0\right)^2}\exp\left[-\frac{1}{N_0}\left(r_1^2+r_2^2+E\right)\right]I_0\left(\frac{2\sqrt{E}}{N_0}r_1\right) \quad (25)$$

When ω_2 is transmitted, the pdf becomes:

$$f(r_1,\psi_1,r_2,\psi_2 \mid \omega_2)=\frac{r_1 r_2}{\left(\pi N_0\right)^2}\exp\left[-\frac{1}{N_0}\left(r_1^2+r_2^2+E\right)\right]I_0\left(\frac{2\sqrt{E}}{N_0}r_2\right) \quad (26)$$

The maximum likelihood decision rule is then:

"Decide ω_1 if $I_0\left(\frac{2\sqrt{E}}{N_0}r_1\right)>I_0\left(\frac{2\sqrt{E}}{N_0}r_2\right)$, decide ω_2 otherwise."

The function I_0 is a monotone increasing function of its argument. The decision rule can be simplified:

"Decide ω_1 if $r_1 > r_2$, decide ω_2 otherwise."

The receiver should compute r_1 and r_2 and compare them.

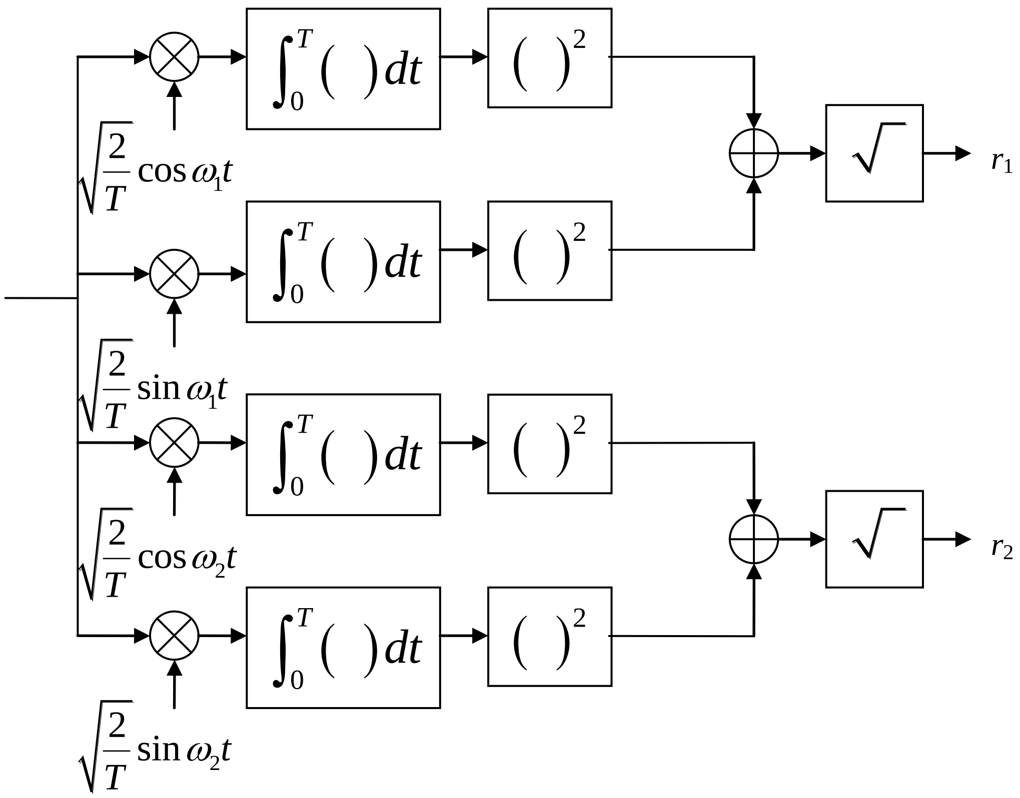


Fig.3- 8 Non coherent FSK receiver

We can show that the above structure can be implemented using two bandpass matched filters tuned respectively to ω_1 and ω_2 followed by envelop detectors. The impulse response of the bandpass filter matched to ω_1 is $h(t) = p(t)\cos(\omega_1 t + \theta)$ where $p(t) = 1$ for $0 \le t \le T$ and zero elsewhere and θ is arbitrary.

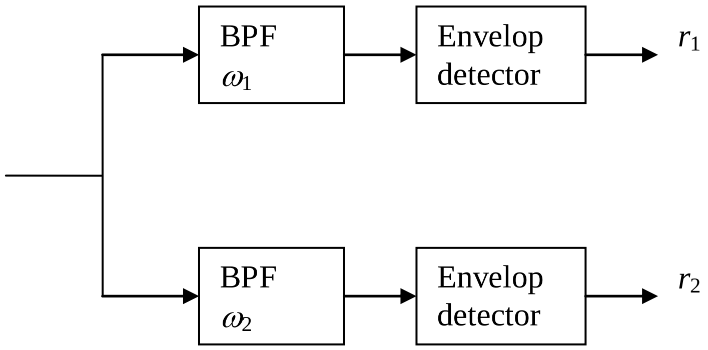
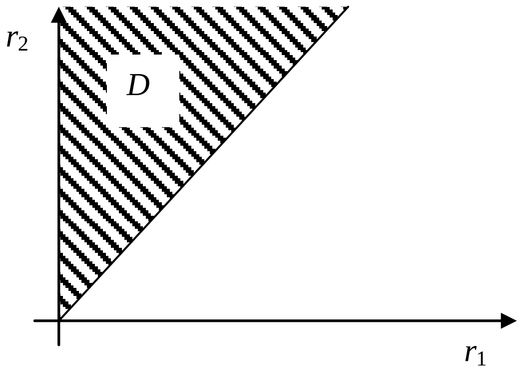


Fig.3- 9 Non coherent FSK receiver

The decision depends only on the amplitude of the output of the bandpass filters. The conditional probability of error can be computed as follows:

Given that ω1 is transmitted, we have an error if r_2 is larger than r_1. So:

$$P(E \mid \omega_1) = \iint_D f(r_1, r_2 \mid \omega_1)\, dr_1 dr_2$$



We need of course to compute the joint pdf of the two envelops r_1 and r_2.

$$f\left(r_1, r_2 \mid \omega_1\right) = \int_0^{2\pi}\int_0^{2\pi} f\left(r_1, \psi_1, r_2, \psi_2\right) d\psi_1 d\psi_2$$

$$= \int_0^{2\pi}\int_0^{2\pi} \frac{r_1 r_2}{\left(\pi N_0\right)^2} \exp\left[-\frac{r_1^2 + r_2^2 + E}{N_0}\right] I_0\left(\frac{2\sqrt{E}}{N_0} r_1\right) d\psi_1 d\psi_2$$

Finally:

$$f\left(r_1, r_2 \mid \omega_1\right) = \frac{4 r_1 r_2}{N_0^2} \exp\left[-\frac{r_1^2 + r_2^2 + E}{N_0}\right] I_0\left(\frac{2\sqrt{E}}{N_0} r_1\right) \qquad (27)$$

From the above expression, we can see that the two variables (r_1 and r_2) are independent and their pdf's are:

$$f\left(r_1 \mid \omega_1\right) = \frac{2 r_1}{N_0} \exp\left[-\frac{r_1^2 + E}{N_0}\right] I_0\left(\frac{2\sqrt{E}}{N_0} r_1\right) \qquad (28)$$

This means that r_1 is Rician.

$$f\left(r_2 \mid \omega_1\right) = \frac{2 r_2}{N_0} \exp\left[-\frac{r_2^2}{N_0}\right] \qquad (29)$$

This means that r_2 is Rayleigh.

The conditional probability of error is:

$$P\left(E \mid \omega_1\right) = \int_0^{+\infty}\left[\int_{r_1}^{+\infty} f(r_1 \mid \omega_1) f(r_2 \mid \omega_2) dr_2\right] dr_1$$

After some simple manipulations, we obtain:

$$P\left(E \mid \omega_1\right) = \frac{1}{2}\exp\left[-\frac{E}{2N_0}\right]$$

The other probability of error has the same expression. Since the two symbols are equiprobable, the probability of error has the same expression also.

$$P\left(E\right) = \frac{1}{2}\exp\left[-\frac{E}{2N_0}\right]$$

1.8 Bit by bit or vertices of a hypercube

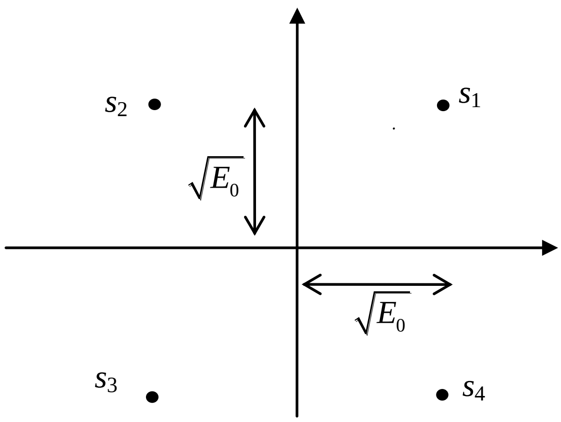
Let us consider messages of length N binary digits. We have $M = 2^N$ different messages. Consider also the following signalling scheme:

To the message m_i, we assign the signal:

$$s_i(t) = \sqrt{E_0}\sum_{k=1}^{N}\alpha_{ik}\varphi_k(t) \qquad ; \qquad 0 \le t \le T \tag{30}$$

where $\alpha_{ik} = +1$ when the k^{th} bit of m_i is 1 and $\alpha_{ik} = -1$ when the k^{th} bit of m_i is 0 and $\varphi_k(t)$ form an orthonormal basis of size N.

For $N = 2$, we obtain the following constellation:



It is evident that the decision regions borders in the above figure are the axis. For the general case, the decision region for $m_1 = (1\ 1\ 1\ \ldots\ 1)$ is the region defined by $[0, +\infty[$ in each dimension.

For an additive white noise with psd equal to $N_0/2$ and m_1, the conditional pdf of the received vector $\mathbf{z}$ is:

$$f(\mathbf{z} \mid m_1) = \left(\frac{1}{\sqrt{\pi N_0}}\right)^N \exp\left[-\frac{1}{N_0}\sum_{k=1}^{N}\left(z_k - \sqrt{N_0}\right)^2\right] \tag{31}$$

From the previous result, the probability of correct decision given m_1 will be the integral of the above pdf over the previously defined decision region.

$$\Pr\left[C \mid m_1\right] = \int_0^{+\infty}\int_0^{+\infty}\cdots\int_0^{+\infty} f\left(\mathbf{z} \mid m_1\right) dz_1 dz_2 \ldots dz_N$$

$$= \prod_{k=1}^{N}\int_0^{+\infty} f(z_k \mid m_1) dz_k \tag{32}$$

where $f(z_k \mid m_1) = \dfrac{1}{\sqrt{\pi N_0}} \exp\left[-\dfrac{1}{N_0}\left(z_k - \sqrt{E_0} \right)^2 \right]$.

Since all decision regions have the same geometry, all the conditional probabilities of correct decision are identical. So, the integral of the above pdf from zero to infinity is:

$$\int_0^{+\infty} f(z_k \mid m_1) dz_k = 1 - p_e = 1 - \frac{1}{2}\operatorname{erfc}\sqrt{\frac{E_0}{N_0}} \tag{33}$$

where p_e is the probability of error for every binary digit of the N bit message. Finally, the probability of error of the above scheme is:

$$\Pr(E) = 1 - \left(1 - p_e\right)^N \tag{34}$$

The result (34) corresponds to bit by bit signaling if we define the basis functions as a function that is different from zero only in the interval $[(k-1)T, kT]$ corresponding to the k^{th} transmitted bit (of duration T). In other words, we transmit one bit every T seconds using antipodal signaling.

If we consider the above result, it seems that it is impossible to transmit correctly a very long message. We are going to see that it is not true.

1.9 The union bound

In the case of a simple geometry of the signal space, it is possible to calculate exactly the probability of error corresponding to a given constellation. In general, if the boundaries of the decision regions are parallel to the axis, it is always possible to have a closed expression of the probability of error (using the erfc function). However, for a general distributions of points in the signal space, this is not the case (the regions of integrations have oblique boundaries).

In the derivations that follow, we are going to assume that we have M equiprobable symbols that can be encoded in binary. This means that we assume also that $M = 2^N$. In the equiprobable case, the boundaries are hyperplanes that separate the segment joining two points of the constellation in its middle. The following figure shows this situation in two dimensions.

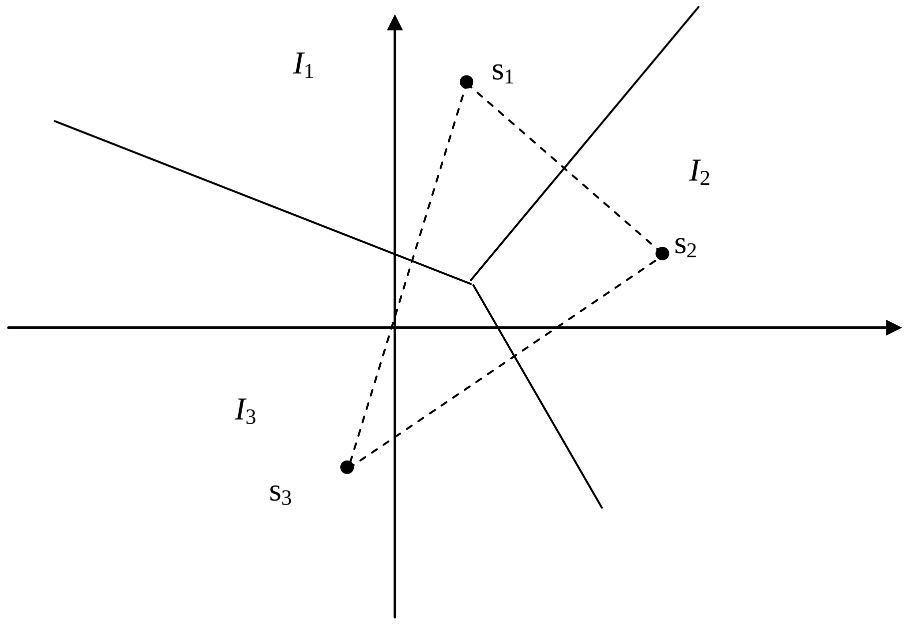


Figure 1

In Figure 1, the plane is divided into three decision regions I_1, I_2 and I_3. The conditional probability of error is given by:

$$\Pr(E \mid m_i) = 1 - \Pr(C \mid m_i) = 1 - \int_{I_i} f(\mathbf{z} \mid m_i)\, d\mathbf{z} = \int_{\bigcup_{k=1, k \neq i}^{M} I_k} f(\mathbf{z} \mid m_i)\, d\mathbf{z} \qquad (35)$$

and the probability of error is then:

$$\Pr(E) = \frac{1}{M} \sum_{i=1}^{M} \Pr(E \mid m_i) \qquad (36)$$

Even in the quite simple case depicted by figure 1, the calculation of the above probability is very difficult. We can sometimes simplify this computation if we translate and/or rotate the figure. It is not hard to show that the above probability evaluations are invariant to translations and rotations. Let us first evaluate the probability of error for the binary case.

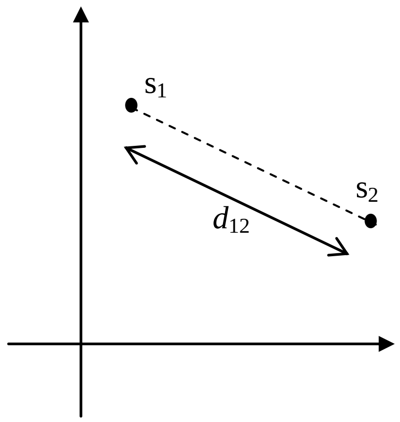


Figure 2

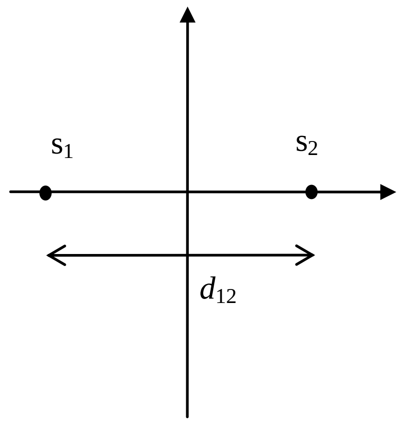


Figure 3

Let us consider the two points separated by distance d_{12} as shown in Figure 2. After a proper translation and rotation, we obtain the configuration of Figure 3. There, the coordinates of the two points are: $\left(-\frac{d_{12}}{2},0\right)$ and $\left(\frac{d_{12}}{2},0\right)$. For a transmission in additive white noise with psd $\frac{N_0}{2}$, the probability of a binary error is

$$\Pr\left[binE\right]=\frac{1}{2}\operatorname{erfc}\left(\frac{d_{12}}{2\sqrt{N_0}}\right) \tag{37}$$

For a more general constellation of points, we can compute an upper bound for the probability of error using the union bound.

The union bound

In a general probability space, if an event is the union of many events:

$$B=\bigcup_{k=1}^{N}A_k \text{ then } \Pr\left(B\right)\le\sum_{k=1}^{N}\Pr\left(A_k\right)$$

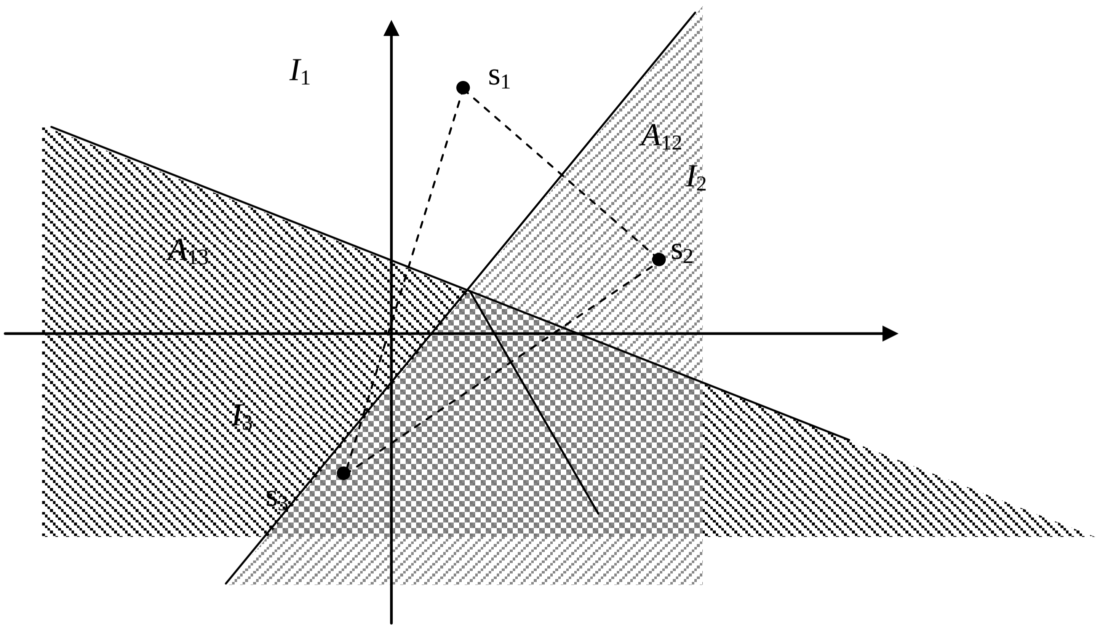


Figure 4

If we come back to the configuration of Figure 1, we see clearly that the event $\{Error \mid m_1\}$ is the union of the two events $\{Binary\ Error(m_1, m_2) \mid m_1\}$ and $\{Binary\ Error(m_1, m_3) \mid m_1\}$ (see Figure 4). For a more general constellation, we can always write:

$$\{Error \mid m_i\} = \bigcup_{\substack{k=1 \\ k \neq i}}^{M} \{Binary\ Error(m_i, m_k) \mid m_i\}$$

Using the union bound, we obtain:

$$\Pr(E \mid m_i) \leq \sum_{\substack{k=1 \\ k \neq i}}^{M} \Pr(binE(m_i, m_k) \mid m_i) \tag{38}$$

Equation(37) provides:

$$\Pr(E \mid m_i) \leq \sum_{\substack{k=1 \\ k \neq i}}^{M} \frac{1}{2} \operatorname{erfc}\left(\frac{d_{ik}}{2\sqrt{N_0}} \right)$$

And finally:

$$\Pr(E) \leq \frac{1}{M} \sum_{i=1}^{M} \sum_{\substack{k=1 \\ k \neq i}}^{M} \frac{1}{2} \operatorname{erfc}\left(\frac{d_{ik}}{2\sqrt{N_0}} \right) \tag{39}$$

In order to use equation(39), we must have all the distances between the different points in the constellation. We can have a more useful bound if we use the smallest distance between points. So, let $d_0 = \min_{\substack{1 \le i,k \le M \\ i \ne k}} d_{ik}$, since $\operatorname{erfc}\left(\frac{d_{ik}}{2\sqrt{N_0}}\right) \le \operatorname{erfc}\left(\frac{d_0}{2\sqrt{N_0}}\right)$, we obtain the final result:

$$\Pr(E) \le \frac{M-1}{2}\operatorname{erfc}\left(\frac{d_0}{2\sqrt{N_0}}\right) \tag{40}$$

Orthogonal signaling

In this section, we are going to use the above derived union bound to show an important result in communication theory. For this signaling, we are going to use the following signal set. To the M symbols m_i, we associate M signals $s_i(t)$ such that:

$$(\mathbf{s}_i, \mathbf{s}_j) = \begin{cases} E & i = j \\ 0 & i \ne j \end{cases}$$

It is clear that the dimension of the signal space is $M = 2^N$. So, the orthonormal basis is formed by the signals s_i scaled to unit energy.

$$\varphi_i(t) = \frac{s_i(t)}{\sqrt{E}}$$

This means that every point in the constellation is $\mathbf{s}_i = \begin{pmatrix} 0 & 0 & \cdots & 0 & \sqrt{E} & 0 & \cdots & 0 \end{pmatrix}^T$, the non zero coordinate corresponds to the i^{th} position. The structure of the optimum receiver is very simple. We simply compute the M dimensional vector $\mathbf{z}$ and the detection process amounts to giving the address of the largest coordinate of $\mathbf{z}$. We can also use the union bound to provide an upper bound for the probability of error. The distance between any two distinct points in the constellation is $d_{ik} = d_0 = \|\mathbf{s}_i - \mathbf{s}_k\| = \sqrt{2E}$. The union bound is:

$$\Pr(E) \le \frac{M-1}{2}\operatorname{erfc}\left(\sqrt{\frac{E}{2N_0}}\right) \tag{41}$$

We introduce the bit energy $E_b = \frac{E}{N}$ and the fact that $M = 2^N$. Equation (41)becomes:

$$\Pr(E) \le \frac{2^N - 1}{2}\operatorname{erfc}\left(\sqrt{\frac{NE_b}{2N_0}}\right)$$

We want to study the behavior of the probability of error for large N. We can use the following asymptotic approximation of the complementary error function:

$$\text{erfc}(x) \approx \frac{\exp(-x^2)}{x\sqrt{\pi}} \quad \text{for } x > 3 \tag{42}$$

Since we consider large N, then $2^N - 1 \approx 2^N = \exp(N \ln 2)$ will also be used in (41).

$$\Pr(E) \le \frac{1}{2}\sqrt{\frac{2N_0}{NE_b}} \exp(N \ln 2) \exp\left(-\frac{NE_b}{2N_0}\right) \tag{43}$$

$$\Pr(E) \le \frac{1}{2}\sqrt{\frac{2N_0}{NE_b}} \exp\left[-N\left(\frac{E_b}{2N_0} - \ln 2\right)\right] \tag{44}$$

We see that if $\frac{E_b}{N_0} > 2\ln 2$, then the probability of error will decrease for large N! This result contradicts the one we have obtained for the bit by bit signaling. The union bound is a very loose bound. This means that the true value of the probability of error can be quite far from the bound. For the orthogonal signaling scheme, we cannot have a closed analytic formulation of the probability of error. However, we can compute it numerically using the following expression (for the derivation, see Ziemer, R.E. and W.H. Tranter, *Principles of communications: systems, modulation and noise*, 6^{th} Ed., John Wiley, 2009):

$$\Pr(E) = 1 - (\pi)^{-M/2} \int_{-\infty}^{+\infty} \exp(-y^2) \left[\int_{-\infty}^{\sqrt{E/N_0}+y} \exp(-x^2)\, dx\right]^{M-1} dy \tag{45}$$

The above expression can be evaluated numerically for various values of M. The result is plotted below.

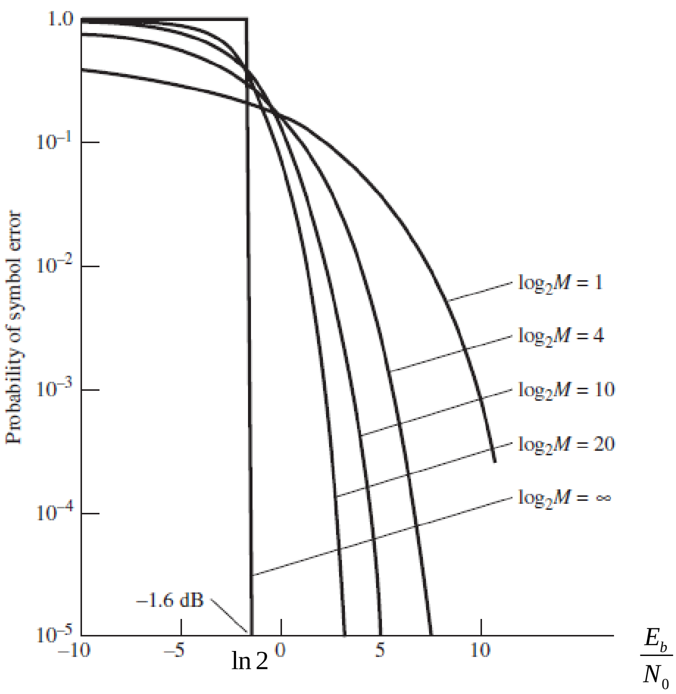


Figure 5 (from same reference)

From Figure 5, we see clearly that, for very long messages ($N = \log_2 M \rightarrow \infty$), we have the following behavior for the probability of error:

$$\Pr(E) \rightarrow 0 \quad \text{if} \quad \frac{E_b}{N_0} > \ln 2 \text{ and } \Pr(E) \rightarrow 1 \quad \text{if} \quad \frac{E_b}{N_0} < \ln 2 \tag{46}$$

There is a threshold for the signal to noise ratio. The above result shows also that if we operate above threshold, we have a great improvement in the quality of communication. However, if we operate below threshold, the system becomes useless. We see also that the threshold predicted by the union bound is too large. In fact, we will see that the result given by (46) is very general and it will be derived by a completely different mechanism using information theory.

We must also remark that the probability of error derived above is a symbol probability of error. However, when we test a given communication system, we usually measure the Bit Error Rate (BER) by counting the number of discrepancies between a known transmitted long message and the received one. For the case of orthogonal signaling, every symbol is encoded with N bits. The only requirement for the encoding is that the assigned codes for different symbols must be different. In orthogonal signaling, the signals associated with the symbols are all equivalent. For this particular case, we can obtain an exact relationship between the probability of bit error and the one of symbol error.