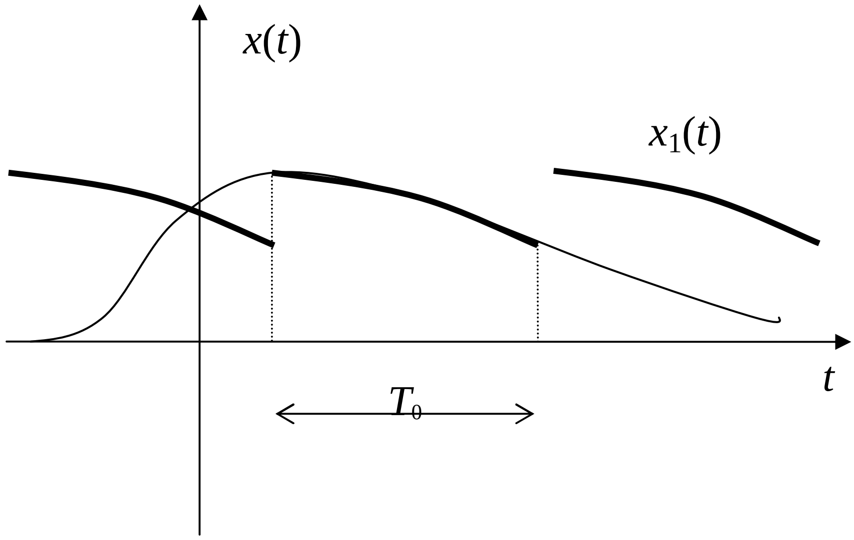
The Fourier Transform.

Consider an energy signal $x(t)$. Its energy is $E = \int_{-\infty}^{+\infty} |x(t)|^2 \, dt$



Such signal is neither finite time nor periodic. This means that we cannot define a "spectrum" for it using Fourier series. In order to try to define a spectrum, let us consider the following periodic signal $x_1(t)$: $x_1(t) = x(t)$ for t inside an interval of width T_0. The signal repeats itself outside of this interval. We can define a Fourier series for this signal. Furthermore, it is evident that:

$$x(t) = \lim_{T_0 \to \infty} x_1(t)$$

We can write:

$$\begin{aligned} x_1(t) &= \sum_{n=-\infty}^{+\infty} c_n e^{jn\omega_0 t} \\ &= \sum_{n=-\infty}^{+\infty} \left[\frac{1}{T_0} \int_{T_0} x(t) e^{-jn\omega_0 t} dt \right] e^{jn\omega_0 t} \end{aligned}$$

As the period T_0 goes to infinity, the spacing between the spectral lines decreases. This spacing is $\Delta\omega = (n+1)\omega_0 - n\omega_0 = \omega_0 = \dfrac{2\pi}{T_0}$. Since this value becomes small, the successive values of the frequencies $n\omega_0$ can be replaced by the continuous variable ω. So, we can rewrite the above expression as:

$$x_1(t) = \frac{1}{2\pi}\sum_{n=-\infty}^{+\infty}\left[\int_{T_0} x(t)e^{-j\omega t}dt\right]e^{j\omega t}\Delta\omega$$

So, when we go to the limit $T_0 \to \infty$, the summation will become an integral and the spacing $\Delta\omega$ will become a differential $d\omega$. So, we obtain:

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\left[\int_{-\infty}^{+\infty} x(t)e^{-j\omega t}dt\right]e^{j\omega t}d\omega$$

The expression between brackets is called the Fourier Transform of the signal $x(t)$. It is a function of the variable ω, so we can write:

$$X(\omega) = \int_{-\infty}^{+\infty} x(t)e^{-j\omega t}dt$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(\omega)e^{j\omega t}d\omega$$

The first expression is called the Forward Fourier Transform or the Analysis Formula. The second one expresses the signal as linear combination of phasors. It is called the Inverse Fourier Transform or the Synthesis Formula. The phasors now have frequencies that belong to a continuum of values. This is why the synthesis formula is now given by an integral and not by a summation. The integrals are computed over infinite intervals. This implies that we have to take into account convergence conditions. Without going into deep mathematical derivation, we can affirm that one sufficient condition for the existence of the Fourier transform is the fact that the signal is an energy signal.

In our course, we will find it easier to use the variable f rather than the variable ω. Since, $\omega = 2\pi f$, we obtain the following pair:

$$X(f) = \int_{-\infty}^{+\infty} x(t) e^{-j2\pi ft} dt$$

$$x(t) = \int_{-\infty}^{+\infty} X(f) e^{j2\pi ft} df$$

The above relations are more interesting than the first ones. They differ by only the sign of the exponent inside the integral.

Symmetry relations:

The Fourier transform $X(f) = \mathfrak{F}[x(t)]$ of the signal $x(t)$ is in general a complex function of the real variable f. We can express it as:

$X(f) = |X(f)| e^{j\varphi(f)}$, where the two functions $|X(f)|$ and $\varphi(f)$ have no particular symmetry if $x(t)$ is complex. However, if x(t) is real, then the Fourier transform will have the same type of symmetry as the one we have seen in the study of Fourier series (Hermitian Symmetry).

So: $X(f) = X^*(-f)$. This means that:

$$|X(f)| = |X(-f)| \quad \text{(Even function)}$$

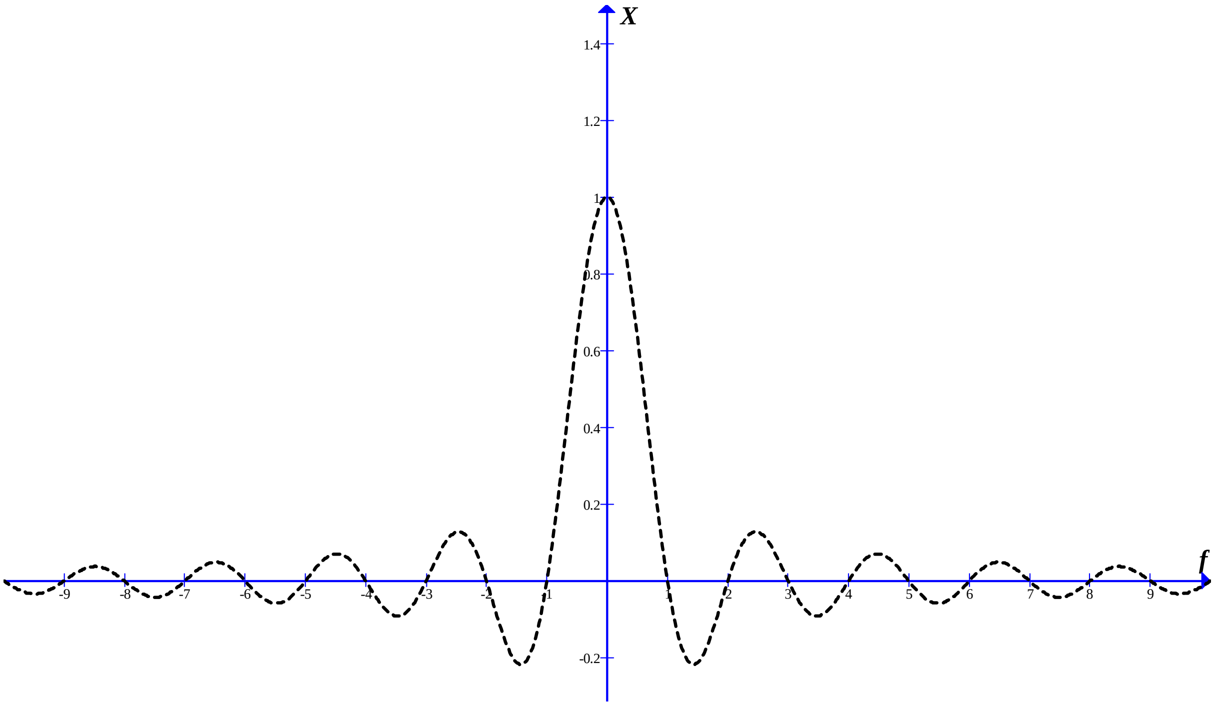
$$\varphi(f) = -\varphi(-f) \quad \text{(Odd function)}$$

Example: The rectangular pulse.

Consider the signal $x(t) = \Pi(t)$.

$$\Pi(t) = \begin{cases} 1 & |t| < 1/2 \\ 0 & |t| > 1/2 \end{cases}$$

We have: $X(f)=\int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-j2\pi ft}dt=\frac{1}{\pi f}\sin\pi f=\operatorname{sinc} f$



The Sinc function

Causal exponential pulse:

$$x(t)=\begin{cases} Ae^{-bt} & t>0 \\ 0 & t<0 \end{cases}$$

$$X(f)=A\int_{0}^{+\infty} e^{-bt}e^{-j2\pi ft}dt=\frac{A}{b+j2\pi f}$$

When the function is causal, the Fourier transform can be seen as the evaluation of the Laplace transform on the imaginary ($j\omega$) axis.

<u>Properties of the Fourier transform:</u>

1. Linearity: $\mathfrak{F}[ax_1(t)+bx_2(t)] = a\mathfrak{F}[x_1(t)] + b\mathfrak{F}[x_2(t)]$

2. Duality: If $x(t)$ and $X(f)$ constitute a known transform pair, then $\mathfrak{F}[X(t)] = x(-f)$. For example: $\mathfrak{F}[\Pi(t)] = \text{sinc}f$. Then $\mathfrak{F}[\text{sinc}t] = \Pi(f)$.

3. Time delay: $\mathfrak{F}[x(t-\tau)] = X(f)e^{-j2\pi f\tau}$

4. Scale change: $\mathfrak{F}[x(at)] = \frac{1}{|a|}X\left(\frac{f}{a}\right)$. So, compressing a signal expands its spectrum and vice versa.

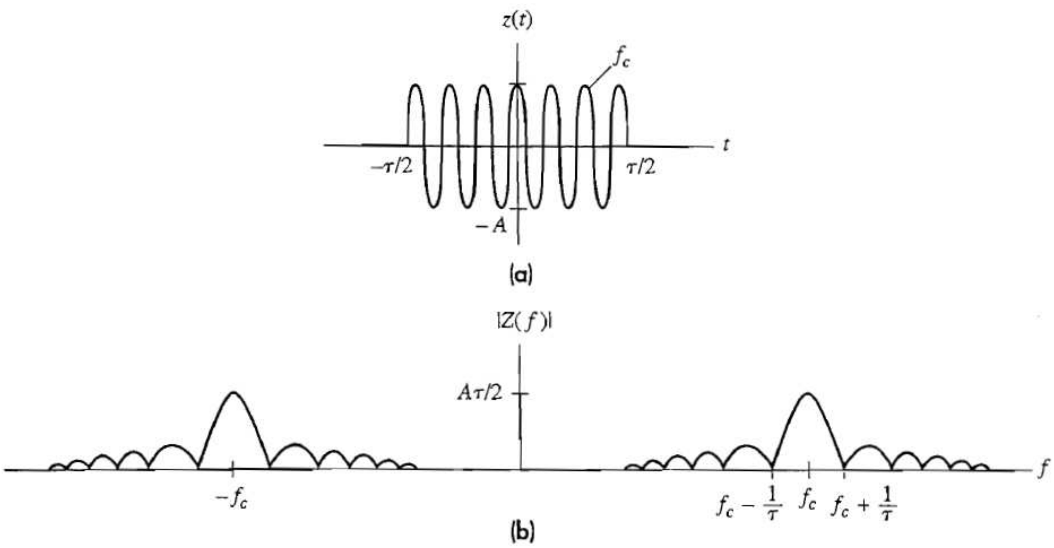
5. Frequency translation: $\mathcal{F}\left[x(t)e^{j\omega_0 t}\right] = X(f-f_0) \qquad \omega_0 = 2\pi f_0$

6. Modulation theorem:

$$\mathcal{F}[x(t)\cos\omega_0 t] = \frac{1}{2}X(f-f_0) + \frac{1}{2}X(f+f_0)$$

Example: The RF pulse: Consider $x(t) = A\Pi\left(\frac{t}{\tau}\right)\cos\omega_0 t$. Using the above relations, we obtain:

$$X(f) = \frac{A\tau}{2}\text{sinc}(f-f_0)\tau + \frac{A\tau}{2}\text{sinc}(f+f_0)\tau$$



7. Differentiation: $\mathcal{F}\left[\frac{d}{dt}x(t)\right] = j2\pi fX(f)$

8. Convolution:

The convolution of two functions is defined as:

$z(t) = x(t) * y(t) = \int_{-\infty}^{+\infty} x(\lambda)y(t-\lambda)d\lambda$. We obtain:

$$\mathcal{F}[x(t) * y(t)] = X(f)Y(f)$$

Example: Consider the triangular function $\Lambda(t)$.

$$\Lambda(t) = \begin{cases} 1-|t| & -1 < t < 1 \\ 0 & \text{elsewhere} \end{cases}$$

It is easy to show that $\Lambda(t) = \Pi(t)*\Pi(t)$

So $\mathcal{F}[\Lambda(t)] = \text{sinc}^2 f$

9. Multiplication:

$$F[x(t)y(t)] = X(f)*Y(f) = \int_{-\infty}^{+\infty} X(\lambda)Y(f-\lambda)d\lambda$$

10. The Rayleigh Energy theorem:

$$\int_{-\infty}^{+\infty} |x(t)|^2 dt = \int_{-\infty}^{+\infty} |X(f)|^2 df$$

The Dirac Impulse Function

The Dirac impulse function or unit impulse function or simply the delta function $\delta(t)$ is not a function in the strict mathematical sense. It is defined in advanced texts and courses using the theory of distributions. In our course, we will suffice with the following much simpler definition.

$$\int_a^b x(t)\delta(t)dt = \begin{cases} x(0) & a < 0 < b \\ 0 & \text{otherwise} \end{cases}$$

where $x(t)$ is an ordinary function that is continuous at $t = 0$.

If $x(t) = 1$, the above expression implies:

$$\int_{-\infty}^{+\infty} \delta(t)dt = \int_{-\varepsilon}^{\varepsilon} \delta(t)dt = 1 \text{ for } \forall \varepsilon > 0$$

We can interpret the above result by saying that the impulse function has a unit area concentrated at the point $t = 0$. Furthermore, we can deduce from the above that $\delta(t) = 0$ for $t \neq 0$. This also means that the delta function is an even function.

The defining integral can also be used to compute the following integral:

$$\int_{-\infty}^{t} \delta(u)du = \begin{cases} 1 & t > 0 \\ 0 & t < 0 \end{cases}$$

The above is nothing but the definition of the unit step function or the Heaviside function. So, we obtain the following relationship between the two functions:

$$u(t) = \int_{-\infty}^{t} \delta(\lambda)d\lambda$$

$$\text{and } \delta(t) = \frac{du(t)}{dt}$$

<u>Properties of the delta function:</u>

1. Replication:

$$x(t) * \delta(t-\tau) = \int_{-\infty}^{+\infty} \delta(\lambda-\tau)x(t-\lambda)d\lambda = x(t-\tau)$$

2. Sifting:

$$\int_{-\infty}^{+\infty} x(t)\delta(t-\tau)dt = x(\tau)$$

3. We can use the fact that the delta function is even in the above integral to show that: $\int_{-\infty}^{+\infty} x(\lambda)\delta(t-\lambda)d\lambda = x(t)$

The above relation is a convolution.

4. Since the expressions containing the impulse function must be integrated, the following properties can be easily deduced:

$$x(t)\delta(t-t_0) = x(t_0)\delta(t-t_0)$$

$$\text{and } \delta(at) = \frac{1}{|a|}\delta(t) \qquad a \neq 0$$

Fourier Transform of the impulse:

$$\int_{-\infty}^{+\infty} \delta(t)e^{-j2\pi ft}dt = 1$$

Using the duality property, we deduce that:

$\mathcal{F}[1] = \int_{-\infty}^{+\infty} e^{-j2\pi ft}dt = \delta(f)$ even though the constant (dc) is a power and not an energy signal. In fact, using the frequency translation property, we can compute the Fourier transform of the phasor:

$$\mathcal{F}\left[e^{j\omega_0 t}\right] = \delta(f - f_0)$$

This allows us to compute the Fourier transform of periodic signals.

If $x(t)$ is periodic with fundamental period T_0, we can develop it in Fourier series: $x(t) = \sum_{n=-\infty}^{+\infty} c_n e^{j2\pi n f_0 t}$. Using the above, we obtain:

$$\mathcal{F}[x(t)] = \mathcal{F}\left[\sum_{n=-\infty}^{+\infty} c_n e^{j2\pi n f_0 t}\right] = \sum_{n=-\infty}^{+\infty} c_n \delta(f - nf_0)$$

Example:

Consider the following impulse train:

$x(t) = \sum_{n=-\infty}^{+\infty} \delta(t - nT_0)$. This function is a repetition of the delta function every T_0 seconds. In the interval $\left[-\frac{T_0}{2}, \frac{T_0}{2}\right]$, we have $x(t) = \delta(t)$ and it is periodic. Its Fourier series coefficients are:

$$c_n = \frac{1}{T_0} \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} \delta(t) e^{-jn\omega_0 t} dt = \frac{1}{T_0}$$

This implies that the Fourier transform of the impulse train is:

$$X(f) = \frac{1}{T_0} \sum_{n=-\infty}^{+\infty} \delta(f - nf_0) = f_0 \sum_{n=-\infty}^{+\infty} \delta(f - nf_0)$$

From these relations, we can now relate Fourier transforms and Fourier series. In order to do so, let us consider the signal $s(t)$ built by a repetition of the signal $x(t)$.

$s(t) = \sum_{k=-\infty}^{+\infty} x(t - kT_0)$ where the signal $x(t)$ has a finite duration T_0. In other words, $x(t) = 0$ for $t \notin [-T_0/2, T_0/2]$. Let $X(f)$ be its Fourier transform. Using the replication property of the delta function, we can write:

$$s(t) = x(t) * \sum_{k=-\infty}^{+\infty} \delta(t - kT_0)$$

This means that

$$S(f) = X(f) \mathcal{F}\left[\sum_{k=-\infty}^{+\infty} \delta(t - kT_0)\right] = f_0 X(f) \sum_{k=-\infty}^{+\infty} \delta(f - kf_0)$$

Using now the sampling property, we can re-express the above as:

$$S(f) = f_0 \sum_{k=-\infty}^{+\infty} X(kf_0)\delta(f - kf_0)$$

Example: Fourier transform of a periodic train of rectangular pulses.

Here, $s(t) = A\sum_{k=-\infty}^{+\infty} \Pi\left(\frac{t-kT_0}{\tau}\right)$ where $\tau < T_0$. In our case, the function $x(t)$ is: $x(t) = A\Pi\left(\frac{t}{\tau}\right)$ with a Fourier transform $X(f) = A\tau \operatorname{sinc}(f\tau)$.

So: $$S(f) = Af_0\tau \sum_{n=-\infty}^{+\infty} \operatorname{sinc}(nf_0\tau)\delta(f - nf_0)$$

So, we have found an alternate way to compute the coefficients of the Fourier series of a periodic waveform. In the above example, the coefficients c_n of the development are:

$$c_n = Af_0\tau \operatorname{sinc}(nf_0\tau) = Ad \operatorname{sinc}(nd)$$

where d is the duty cycle $d = f_0\tau = \frac{\tau}{T_0}$.

Fourier transform of the unit step function and of the signum function:

The signum function sgn(t) is a function that is related to the unit step function. It is defined as:

$$\operatorname{sgn}(t) = \begin{cases} 1 & t > 0 \\ 0 & t = 0 \\ -1 & t < 0 \end{cases}$$

It is evident that $u(t)$ = ½sgn(t) + ½. The signum function has zero net area. It can be seen also that sgn(t) is the limit of the following function:

$$z(t) = \begin{cases} e^{-bt} & t > 0 \\ 0 & t = 0 \\ -e^{bt} & t < 0 \end{cases}$$

$b > 0$. We have $\mathcal{F}\left[z(t)\right] = \dfrac{-j4\pi f}{b^2 + \left(2\pi f\right)^2}$. So,

$\mathcal{F}\left[\text{sgn}(t)\right] = \lim\limits_{b \to 0} Z(f) = \dfrac{1}{j\pi f}$. And from the relation between the signum and the unit step function, we get:

$$\mathcal{F}\left[u(t)\right] = \frac{1}{j2\pi f} + \frac{1}{2}\delta(f).$$

By duality, we also obtain: $\mathcal{F}^{-1}\left[\text{sgn}(f)\right] = \dfrac{-1}{j\pi t}$

The unit step function transform allows us to compute the Fourier transform of the integral of a signal.

$$\int_{-\infty}^{t} x(\lambda)d\lambda = \int_{-\infty}^{+\infty} x(\lambda)u(t-\lambda)d\lambda = x(t) * u(t).$$

So, $\mathcal{F}\left[\int_{-\infty}^{t} x(\lambda)d\lambda\right] = \dfrac{X(f)}{j2\pi f} + \dfrac{1}{2}X(0)\delta(f)$

Signals can be classified according to their spectral occupancy. A lowpass (or baseband) signal is a signal with high components at low frequencies and small components at high frequencies. On the other hand, if the spectrum is significantly different from zero only in a band of frequencies all different from zero, the signal is call bandpass.

The width of this band is called the bandwidth. If the ratio of the bandwidth to the value of the center of the band is small, the signal is said to be a narrow bandpass signal.

Bandlimited Signals and the Sampling Theorem

A signal $x(t)$ with Fourier transform $X(f)$ is said to be bandlimited if $X(f) = 0$ for $|f| > W$. The frequency W is the bandwidth of the signal.

Bandlimited signals have the property to be uniquely represented by a sequence of their values obtained by uniformly sampling the signal.

So, to a signal $x(t)$, we can associate a sequence $x_1(n) = x(nT_s)$. T_s is called the sampling period. Its inverse f_s is the sampling frequency.

The Sampling Theorem:

Given a bandlimited *signal x(t) with spectrum X(f) = 0 for |f | > W. The signal can be recovered from its samples x(nT*$_s$*) taken at a rate* $f_s = 1/T_s$ *with* $f_s \geq 2W$.

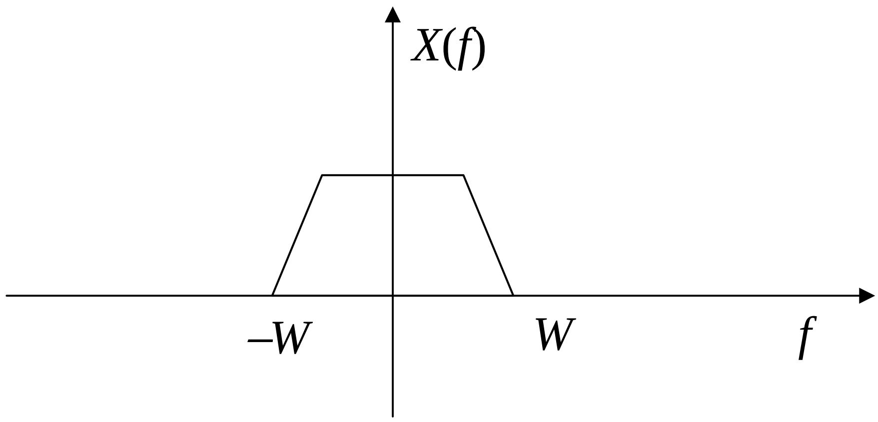
$$x(t) = \sum_{n=-\infty}^{+\infty} x(nT_s)\,\text{sinc}\left(\frac{t - nT_s}{T_s}\right)$$

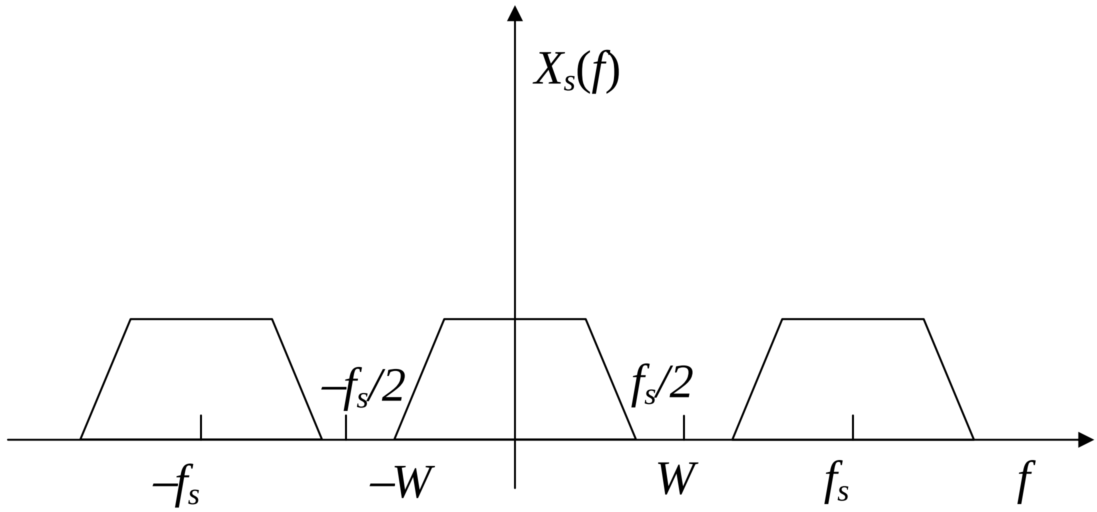
Proof:

We have seen that $x(t)\delta(t - nT_s) = x(nT_s)\delta(t - nT_s)$. So, if we multiply the signal by the impulse train $\sum_{n=-\infty}^{+\infty} \delta(t - nT_s)$, we obtain the sequence of values $x_1(n) = x(nT_s)$. Let us call the obtained signal $x_s(t)$. $x_s(t) = x(t) \times \sum_{n=-\infty}^{+\infty} \delta(t - nT_s) = \sum_{n=-\infty}^{+\infty} x(nT_s)\delta(t - nT_s)$. The Fourier transform of this signal is obtained by the following convolution:

$$X_s(f) = X(f) * \mathscr{F}\left[\sum_{n=-\infty}^{+\infty} \delta(t - nT_s)\right] = X(f) * \left\{ f_s \sum_{n=-\infty}^{+\infty} \delta(f - nf_s) \right\} \quad .$$

Using the replication property of the delta function, we obtain immediately: $X_s(f) = f_s \sum_{n=-\infty}^{+\infty} X(f - nf_s)$. We see that the spectrum of the signal $x_s(t)$ is a repetition of the spectrum of the signal $x(t)$.





The above figures show the relationship between spectra. It is clear that if $f_s/2 > W$, The spectrum of $x_s(t)$ and the one of $x(t)$ will coincide for the range of frequencies between $-f_s/2$ and $f_s/2$ (within the scale factor f_s). This means that we can recover $x(t)$ by computing the inverse Fourier transform of the spectrum $X_s(f)$ multiplied by a "rectangular" filter with a transfer function $\Pi\left(\frac{f}{f_s}\right)$. So, we have:

$$x(t) = \mathscr{F}^{-1}\left[\frac{1}{f_s} X_s(f)\Pi\left(\frac{f}{f_s}\right)\right]$$

The result is finally (show it):

$$x(t) = \sum_{n=-\infty}^{+\infty} x(nT_s)\,\mathrm{sinc}\left(\frac{t-nT_s}{T_s}\right)$$

(q.e.d)

In the above proof, we had to have $f_s > 2W$. If this does not occur, we have the phenomenon of aliasing. Aliasing is a distortion that cannot be cured (in general). It is due to the superposition of the different shifted spectra. We can observe aliasing when we watch western movies. The wagon wheels seem to rotate in reverse. This is due to the sampling rate (number of images/second) which is too small.

So, sampling is the process of generating a sequence (discrete time signal) from a continuous time one. The obtained sequence can be analyzed in the frequency domain. Let us consider $x_1(n) = x(nT_s)$. Its Fourier transform is defined to be:

$$X_1(\omega) = \sum_{n=-\infty}^{+\infty} x_1(n)e^{-jn\omega}$$

We can remark that this spectrum is periodic (in the frequency domain), with a period equal to 2π. In the definition of the spectrum of a sequence, we can also observe that the frequencies are now measured in radians and not in radians per second as for the continuous time signal. This is due to the fact that, in a sequence, the "time" variable *n* is an integer indicating just the position of the sample and not the time position measured in seconds. You should consult the lab manual in order to have the correspondence between the spectrum of the sequence and the one of the original signal.

If the sequence exists for a finite time, i.e. for *N* samples, then the sum is finite.

$$X_1(\omega) = \sum_{n=0}^{N-1} x_1(n) e^{-jn\omega}$$

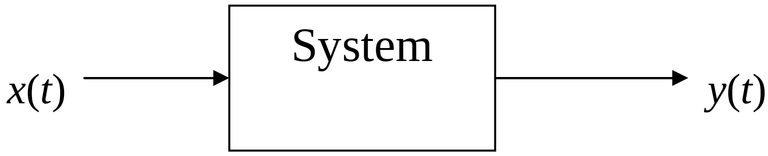
We can also compute the spectrum of the sequence over a finite number of discrete frequencies $\omega_k = \frac{2\pi k}{N}$, $k = 0, \cdots, N-1$.

$$X_1(k) = X_1(\omega_k) = \sum_{n=0}^{N-1} x_1(n) e^{-jn\omega_k} = \sum_{n=0}^{N-1} x_1(n) e^{-j\frac{2\pi}{N}nk}$$

The above relation defines the Discrete Fourier Transform (DFT). This transform can be computed very efficiently using an algorithm called the Fast Fourier Transform (FFT).

Linear Time Invariant Systems

Signals are processed by systems. By the word system, we understand a mapping from a signal set (input signals) to another signal set (output signals).



The above figure shows graphically the relationship that exists between the input signal and the output one. In the mapping, we understand that the whole signal $x(t)$ is transformed into the whole signal $y(t)$. You can encounter the notation: $y(t) = H[x(t)]$. This notation can be misleading. It can also mean that the value of the signal y at the time t is functionally related to only the value of the signal x at t. When we want to indicate the functional relationship between values, we will use the following notation:

$$y(t) = H\left[t, x(\lambda), \lambda \in [t_1, t_2]\right]$$

This means that the value of the output y at the time t depends on all the values of the input signal at times λ between t_1 and t_2 and also on the state of the system at t.

Memoriless system:

If $y(t_0) = H\left[x(t_0)\right]$, i.e. the output at time t_0 depends only on the input at the same time t_0, the system is said to be memoriless.

Causal and anticausal system:

If $y(t) = H\left[x(\lambda), \lambda \le t\right]$, i.e. if the output depends only on the past and on the present (but not on the future), the system is said to be causal.

If $y(t) = H\left[x(\lambda), \lambda > t\right]$, i.e. the system output depends only on the future, the system is called anticausal.

Stable system:

If a bounded input ($|x(t)| \le M, \forall t$) produces a bounded output, we say that the system is BIBO stable.

Linear system:

A system is linear if it satisfies the condition of superposition:

If $y_1(t)$ is the output corresponding to $x_1(t)$ and $y_2(t)$ is the output corresponding to $x_2(t)$, then $a_1y_1(t)+a_2y_2(t)$ corresponds to $a_1x_1(t)+a_2x_2(t)$.

Time Invariant system:

A system is time invariant if it is not affected by a shift of the time origin. In other words, its properties remain the same as time goes by. One consequence is that if $x(t)$ produces $y(t)$, then $x(t-\tau)$ will produce $y(t-\tau)$.

Many systems of interest are linear and time invariant (LTI). Among such systems, we find most of the filters used to select signals in communication systems.

Linear Time Invariant systems:

LTI systems are systems that can be completely described by a single function: the Impulse Response.

If the input of an LTI system is a Dirac impulse, the corresponding output is a function $h(t)$. We have seen that an signal $x(t)$ can be seen a linear combination of shifted delta functions.

$$\int_{-\infty}^{+\infty} x(\lambda)\delta(t-\lambda)d\lambda = x(t)$$

So, since the system is time invariant, then the output corresponding to $\delta(t-\lambda)$ is $h(t-\lambda)$. The system is also linear, so the output corresponding to $x(\lambda)\delta(t-\lambda)$ is $x(\lambda)h(t-\lambda)$. Finally, the output corresponding to $x(t)$ is the sum of such values:

$$y(t) = \int_{-\infty}^{+\infty} x(\lambda)h(t-\lambda)d\lambda$$

The above relation is a convolution. It is easy to show that the convolution is a commutative operation. This means that we can write also:

$$y(t) = \int_{-\infty}^{+\infty} h(\lambda)x(t-\lambda)d\lambda$$

The function $h(t)$ is the impulse response. It describes completely the LTI system and allows us the compute the output for any given input.

We can test the stability of the system by testing its impulse response. A necessary and sufficient condition for a system to be BIBO stable is

$$\int_{-\infty}^{+\infty} |h(t)|\, dt < \infty$$

The above condition also implies that the Fourier transform of a BIBO stable system exists. It is called the transfer function $H(f) = \mathcal{F}[h(t)]$. If the input and output signals possess Fourier transforms, we can write:

$$Y(f) = H(f)X(f)$$

Causality also imposes restrictions on $h(t)$. If the system is causal, the output from the convolution integral should not depend on values of the input at times λ coming after the time t. This implies that $h(t-\lambda) = 0$ for $\lambda > t$. So, we see that in order to have causality, we must have $h(t) = 0$ for $t < 0$. So, when a system is causal, the input output relation becomes:

$$y(t) = \int_{-\infty}^{t} x(\lambda) h(t-\lambda) d\lambda = \int_{0}^{+\infty} h(\lambda) x(t-\lambda) d\lambda$$

<u>Response of an LTI system to a phasor and to a sinewave:</u>

Let the input of the LTI system be the phasor $Ae^{j(\omega_0 t+\theta)}$. The output will be:

$$y(t) = \int_{-\infty}^{+\infty} h(\lambda) A e^{j(\omega_0 (t-\lambda)+\theta)} d\lambda = A e^{j(\omega_0 t+\theta)} \int_{-\infty}^{+\infty} h(\lambda) e^{-j\omega_0 \lambda} d\lambda$$

Replacing $\omega_0 = 2\pi f_0$, we recognize the Fourier transform of the impulse response, the transfer function $H(f_0)$. So

$$y(t) = H(f_0) A e^{j(\omega_0 t+\theta)}$$

If we introduce the modulus $|H(f_0)|$ and argument $\varphi(f_0)$ of the transfer function:

$$y(t) = |H(f_0)| e^{j\varphi(f_0)} A e^{j(\omega_0 t+\theta)}$$

From the above result, we can conclude two important facts.

1^{st}) the response of a phasor of frequency f_0, is also a phasor. The output phasor is proportional to the input one. The constant of proportionality is the transfer function. Since an LTI system is a linear operator, we can say that the phasors are the "eigenfunctions" of LTI systems while the transfer function is the "eigenvalue".

2^{nd}) the output phasor is equal to the input phasor scaled by the modulus of the transfer function and phase shifted by its argument.

Now, if the input is a sinewave $x(t) = A\cos(\omega_0 t + \theta)$, we can write:

$x(t) = \frac{A}{2} e^{j(\omega_0 t+\theta)} + \frac{A}{2} e^{-j(\omega_0 t+\theta)}$, the output becomes:

$$y(t) = H(f_0)\frac{A}{2}e^{j(\omega_0 t+\theta)} + H(-f_0)\frac{A}{2}e^{-j(\omega_0 t+\theta)}$$ giving

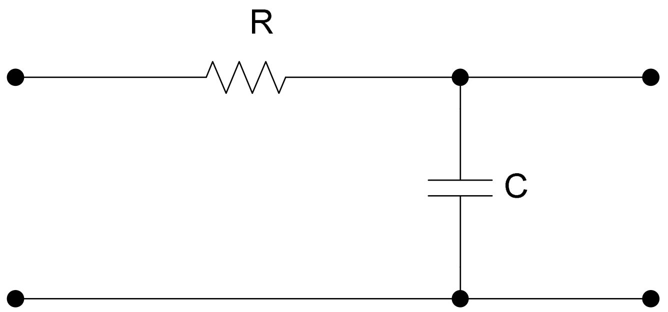
$$y(t) = |H(f_0)|e^{j\varphi(f_0)}\frac{A}{2}e^{j(\omega_0 t+\theta)} + |H(-f_0)|e^{j\varphi(-f_0)}\frac{A}{2}e^{j(\omega_0 t+\theta)}$$. If the impulse response is real, then $|H(f_0)| = |H(-f_0)|$ and $\varphi(f_0) = -\varphi(-f_0)$. This implies:

$$y(t) = A|H(f_0)|\left[\frac{1}{2}e^{j(\omega_0 t+\theta+\varphi(f_0))} + \frac{1}{2}e^{-j(\omega_0 t+\theta+\varphi(f_0))}\right]$$

$$y(t) = A|H(f_0)|\cos\left(\omega_0 t+\theta+\varphi(f_0)\right)$$

The output is a sinewave at the same frequency, scaled by the modulus of the transfer function and phase shifted by the argument of the transfer function at that frequency.

Example: Consider the following RC circuit:



The transfer function is equal to: (this is a simple voltage divider)

$$H(f) = \frac{\frac{1}{j2\pi fC}}{R+\frac{1}{j2\pi fC}} = \frac{1}{1+j2\pi fRC} = \frac{1}{1+j\frac{f}{f_3}}$$

where $f_3 = \frac{1}{2\pi RC}$ is the 3 *dB* cut-off frequency. Assume we input a sinewave at the frequency f_3, $x(t) = A\cos\left(2\pi f_3 t\right)$.

$$\left|H(f_3)\right| = \frac{1}{\sqrt{1+1}} = \frac{\sqrt{2}}{2} \text{ and } \varphi(f_3) = -\tan^{-1} 1 = -\frac{\pi}{4}. \text{ So:}$$

$$y(t) = \frac{A}{\sqrt{2}} \cos\left(2\pi f_3 t - \frac{\pi}{4}\right)$$

When the LTI system is used to modify the spectrum of a signal, it is called a filter. We can classify filters according to their amplitude response. Let $H(f)$ be the transfer function.

If $|H(f)| \cong 0$ for $|f| > W$, the filter is called Lowpass.

If $|H(f)| \cong 0$ for $|f| < W$, the filter is called Highpass.

If $|H(f)| \cong 0$ for $0 < |f| < f_1$ and $|f| > f_2$, the filter is called Bandpass.

If $|H(f)|$ = constant for all frequencies, the filter is an Allpass filter.

Bandpass Signals:

Bandpass signals form an important class of signals. This is due to the fact that practically all methods for transporting information use modulation systems that transform the baseband information into bandpass signals. In this section, we are concerned with real bandpass signals. Let $x(t) \in \mathbb{R}$ be a bandpass signal. Due to the Hermitian symmetry ($X(f) = X^*(-f)$), the information in the positive frequency is enough to characterize completely the signal. A real bandpass signal is characterized by:

$$X(f) = 0 \text{ for } 0 < |f| < f_1 \text{ and } |f| > f_2 \ (f_1 < f_2).$$

The bandwidth of the bandpass signal is defined as being the difference $B = f_2 - f_1$. If $B << f_1$, the signal is a narrow bandpass signal.

Bandpass signals are completely described by their low frequency envelop.

In order to describe quite simply bandpass signals, we have to introduce some mathematical tools.

The Hilbert Transform

In this section of the course, we are going to introduce a tool that allows us to transform a real signal, with a two sided spectrum, into a complex signal with the same spectrum, but only for positive frequencies.

To begin, consider the real sinewave $x(t) = \cos\omega t$ and the phasor $x_+(t) = \exp(j\omega t) = \cos\omega t + j\sin\omega t$. The respective spectra are:

$X(f) = \frac{1}{2}\delta(f - f_0) + \frac{1}{2}\delta(f + f_0)$ and $X_+(f) = \delta(f - f_0)$.

We can remark that the spectrum of the phasor is the same (within a scale factor of 2) as the one of the sinewave for the positive frequencies while it is zero for negative frequencies. Furthermore, the phasor is equal to the sum of the sinewave and the same signal phase shifted by 90°. We can generalize this relationship to most signals.

In order to phase shift signals by 90°, we introduce a transform called the Hilbert Transform.

The Hilbert transform of a signal $x(t)$ is the signal $\hat{x}(t)$ equal to the original signal with all frequencies phase shifted by 90°. The operation of shifting the phase of a signal by a constant value is a linear time invariant operation. This means that the signal $\hat{x}(t)$ is obtained from $x(t)$ by a filtering operation. In fact, the filter is an allpass one. This means that:

$\hat{x}(t) = h(t) * x(t)$ or $\hat{X}(f) = H(f)X(f)$ where $|H(f)| = 1$ and

$$\arg[H(f)] = \begin{cases} -\dfrac{\pi}{2} & f > 0 \\ \dfrac{\pi}{2} & f < 0 \end{cases}$$

So, we can write $H(f) = e^{-j\frac{\pi}{2}\operatorname{sgn} f} = -j\operatorname{sgn} f$.

We have already computed the Fourier transform of the signum function. Using the duality property, we obtain:

$$h(t) = \frac{1}{\pi t}$$

giving

$$\hat{x}(t) = \frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{x(\lambda)}{t-\lambda} d\lambda$$

In general it is easier to compute the Hilbert transform in the frequency domain since it amounts to shifting the frequencies by 90°.

$\widehat{\cos\omega t} = \sin\omega t, \quad \widehat{\sin\omega t} = -\cos\omega t.$

If we apply the Hilbert transform to a signal $\hat{x}(t)$ that is itself the Hilbert transform of a signal $x(t)$, we phase shift $x(t)$ by 180°. This means that we simply invert the signal.

$$\hat{\hat{x}}(t) = -x(t)$$

The above relation provides the inversion formula:

$$x(t) = -\frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{\hat{x}(\lambda)}{t-\lambda} d\lambda$$

The following property is important in the analysis of bandpass signals.

Theorem:

Given a baseband signal $x(t)$ with $X(f) = 0$ for $|f| \geq W$ and a highpass signal $y(t)$ with $Y(f) = 0$ for $|f| < W$ (non-overlapping spectra), then $\widehat{x(t)y(t)} = x(t)\widehat{y(t)}$.

Example: if $y(t) = x(t)\cos\omega_0 t$ with $X(f) = 0$ for $|f| \geq \omega_0$, then $\hat{y}(t) = x(t)\sin\omega_0 t$.

The analytic signal

using the analogy of the sinusoid and the phasor, we can define a signal having a spectrum that exists only for positive frequencies. It is the analytic signal associated with $x(t)$:

$$x_+(t) = x(t) + j\hat{x}(t)$$

We obtain, in the frequency domain:

$$X_+(f) = X(f) + j\hat{X}(f) = X(f)\left[1 + \operatorname{sgn} f\right]$$

so, $$X_+(f) = \begin{cases} 2X(f) & f > 0 \\ X(0) & f = 0 \\ 0 & f < 0 \end{cases}$$

We can also define another analytic signal, but that exists only for negative frequencies.

$$x_-(t) = x(t) - j\hat{x}(t)$$

$$X_-(f)=\begin{cases}2X(f) & f<0\\ X(0) & f=0\\ 0 & f>0\end{cases}$$

We can remark that $x(t)=\mathrm{Re}\left[x_+(t)\right]=\mathrm{Re}\left[x_-(t)\right]$. So, it is simple to extract the original signal from the analytic one. We have also

$x(t)=\dfrac{x_+(t)+x_-(t)}{2}$ which is analogous to $\cos\omega t=\dfrac{e^{j\omega t}+e^{-j\omega t}}{2}$

Using analytic signals, we can now give important properties of bandpass signals.

Consider a real bandpass signal $x(t)$, such that $X(f)=0$ for $0<\left|f\right|<f_1$ and $\left|f\right|>f_2$ ($f_1<f_2$), and consider a frequency f_0 between f_1 and f_2, i.e. $f_1\le f_0\le f_2$, then we can express the signal as:

$$x(t)=a(t)\cos\omega_0 t-b(t)\sin\omega_0 t$$

where $a(t)$ and $b(t)$ are baseband signals bandlimited to $\max\left[f_2-f_0,f_0-f_1\right]$. The above representation is called the quadrature representation. We can also represent the signal as

$$x(t)=r(t)\cos\left(\omega_0 t+\varphi(t)\right)$$

where $r(t)$ and $\varphi(t)$ are also baseband signals. This representation is called a modulus (amplitude) and phase (argument) representation.

Proof:

Consider the analytic signal associated with $x(t)$. $x_+(t)=x(t)+j\hat{x}(t)$. Its spectrum is $X_+(f)=2X(f)$ for positive frequencies and zero for negative ones, i.e. $X_+(f)=2X(f)u(f)$. If we shift its spectrum down to dc by f_0, we obtain a bandlimited signal

$$m_x(t) = x_+(t)e^{-j\omega_0 t}$$

$$M_x(f) = X_+(f + f_0) = 2X(f + f_0)u(f + f_0)$$

This signal is baseband and is in fact bandlimited to $\max\left[f_2 - f_0, f_0 - f_1\right]$. Its spectrum has no particular symmetry (in general). So, the signal is complex. This means that we can write:

$m_x(t) = a(t) + jb(t)$ where the two signals are real and have the same bandwidth as $m_x(t)$. The signal $m_x(t)$ is called the complex envelop of $x(t)$. We can recover the signal $x(t)$ by shifting it back to f_0.

$$x_+(t) = m_x(t)e^{j\omega_0 t}$$

$$\begin{aligned} x(t) &= \mathrm{Re}\left[m_+(t)e^{j\omega_0 t}\right] = \mathrm{Re}\left[\left(a(t) + jb(t)\right)\left(\cos\omega_0 t + j\sin\omega_0 t\right)\right] \\ &= a(t)\cos\omega_0 t - b(t)\sin\omega_0 t \end{aligned}$$

We can also express the complex envelop in modulus and phase:

$m_x(t) = r(t)e^{j\varphi(t)}$ giving: $x(t) = r(t)\cos\left[\omega_0 t + \varphi(t)\right]$ along with

$$r(t) = \sqrt{a^2(t) + b^2(t)},\ \varphi(t) = \tan^{-1}\frac{b(t)}{a(t)}$$

$$a(t) = r(t)\cos\omega_0 t,\ b(t) = r(t)\sin\omega_0 t$$

(q.e.d.)

In a bandpass signal, the information is contained in the complex envelop. In many cases, it is easier to process the envelop of the signal instead of processing directly the bandpass signal.

Filtering a bandpass signal:

Consider a real narrow bandpass signal $x(t)$ having a bandwidth W centered around a frequency f_0 and consider a real bandpass filter (with impulse response $h(t)$)with a bandwidth B that covers completely the signal $x(t)$. The transfer function $H(f)$ is of course the Fourier transform of $h(t)$.

We define the equivalent lowpass filter $h_{lp}(t)$ as the lowpass filter having as transfer function $H_{lp}(f)$ the positive frequency half of $H(f)$ translated down to zero by $-f_0$. So:

$$H_{lp}(f) = H(f + f_0)u(f + f_0)$$

If we call $x(t)$ the input of the bandpass filter and $y(t)$ the output, we have:

$y(t) = h(t) * x(t)$ or $Y(f) = H(f)X(f)$. Introducing the complex envelops:

$x_+(t) = m_x(t)e^{j\omega_0 t}$ and $y_+(t) = m_y(t)e^{j\omega_0 t}$. In the frequency domain, this becomes: $X_+(f) = M_x(f - f_0)$ and $Y_+(f) = M_y(f - f_0)$.

Since $Y(f) = H(f)X(f)$, then $Y_+(f) = H(f)X_+(f)$. This implies that $M_y(f - f_0) = H(f)M_x(f - f_0)$, this relation is valid for positive frequencies, so we can write, without affecting the previous relation:

$M_y(f - f_0)u(f) = H(f)M_x(f - f_0)u(f)$. If we make the change of variable $f' = f - f_0$, we obtain:

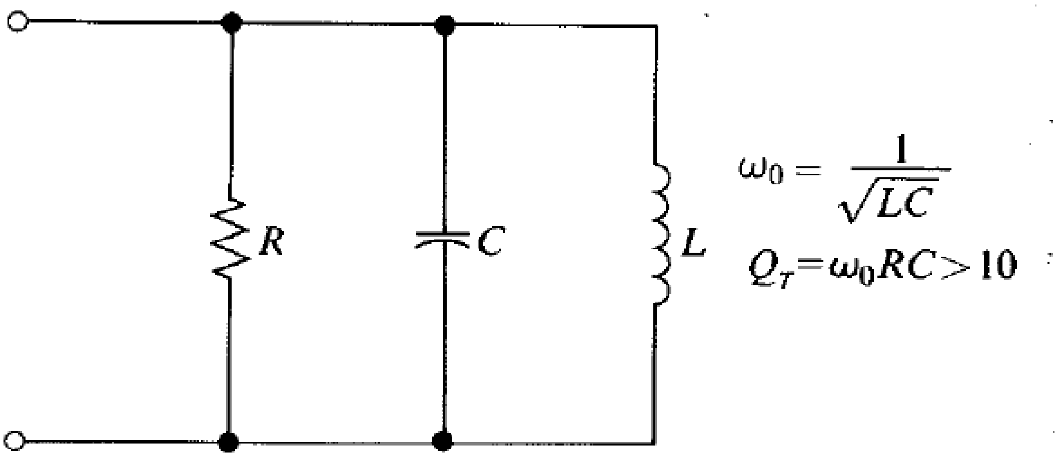
$M_y(f')u(f' + f_0) = H(f' + f_0)u(f' + f_0)M_x(f')$ or

$$M_y(f) = H_{lp}(f)M_x(f)$$

So, bandpass filtering a bandpass signal amounts to lowpass filtering its complex envelop by the equivalent lowpass filter.

Example:

Consider the following parallel RLC circuit:



The impedance of the circuit is:

$$Z(\omega) = \frac{1}{\frac{1}{R} + jC\omega + \frac{1}{jL\omega}} = \frac{R}{1 + jQ_T \frac{\omega^2 - \omega_0^2}{\omega\omega_0}} \quad \text{where} \quad \omega_0 = \frac{1}{\sqrt{LC}} \quad ;$$

$Q_T = RC\omega_0$.

If $Q_T > 10$, this impedance can be approximated quite closely by:

$Z(\omega) = \dfrac{R}{1 + j\dfrac{\omega - \omega_0}{\alpha}}$, $\omega > 0$, $\alpha = \dfrac{1}{2RC}$ and it is essentially different from zero only in the vicinity of ω_0.

If the input is the current flowing through the circuit and the output is the voltage, we have a narrow bandpass filter. Let the current $x(t)$ be:

$x(t) = A\cos\omega_m t\cos\omega_0 t$ with $\omega_m << \omega_0$. The signal is already in quadrature form with $a(t) = A\cos\omega_m t$, $b(t) = 0$. So, the complex envelop is $m_x(t) = A\cos\omega_m t$. The equivalent lowpass filter has is:

$Z_{lp}(f) = \dfrac{R}{1 + j\dfrac{2\pi f}{\alpha}}$ so the complex envelop of the output is the sinewave $m_x(t)$ filtered by $Z_{lp}(f)$. So,

$$m_y(t) = \left|Z_{lp}(f_m)\right| A\cos\left(\omega_m t + Arg\left[Z_{lp}(f_m)\right]\right)$$

So, $m_y(t) = \dfrac{AR}{\sqrt{1 + \left(\dfrac{2\pi f_m}{\alpha}\right)^2}}\cos\left(\omega_m t - \tan^{-1}\left(\dfrac{2\pi f_m}{\alpha}\right)\right)$

And finally

$$y(t) = \mathrm{Re}\left[m_y(t)e^{j\omega_0 t}\right] = \frac{AR}{\sqrt{1 + \left(\dfrac{2\pi f_m}{\alpha}\right)^2}}\cos\left(\omega_m t - \tan^{-1}\left(\frac{2\pi f_m}{\alpha}\right)\right)\cos\omega_0 t$$

Group delay and phase delay:

Consider a very narrow bandpass signal centered on a frequency f_0 and having a bandwidth W ($W << f_0$). This signal is to be filtered by a filter having a transfer function $H(f)$. The signal is:

$$x(t) = r(t)\cos\left[\omega_0 t + \theta(t)\right]$$

Because of the narrowness of the bandwidth of the signal, we can make the following approximations for the transfer function:

$H(f) = A(f)\exp\left[j\varphi(f)\right]$ and around f_0, we can assume that the amplitude response is constant, and that the phase response can be approximated by its first order Taylor series.

$A(f) \simeq A_0$ and $\varphi(f) \simeq \varphi_0 + k(f - f_0)$ for f around f_0. $k = \left.\dfrac{d\varphi}{df}\right|_{f=f_0}$.

The complex envelop of the signal is: $m_x(t) = r(t)\exp\left[j\theta(t)\right]$ and the equivalent lowpass filter transfer function is:

$H_{lp}(f) = H(f + f_0)u(f + f_0) = A_0 \exp\left[j\left(\varphi_0 + kf\right)\right]$.

So, the complex envelop of the output is given by:

$M_y(f) = A_0 e^{j(\varphi_0 + kf)} M_x(f) = A_0 e^{j\varphi_0} M_x(f) e^{jkf}$. Using the time delay theorem: $m_y(t) = A_0 e^{j\varphi_0} m_x\left(t + \dfrac{k}{2\pi}\right)$. Finally, the output of the filter is:

$$y(t) = \mathrm{Re}\left\{A_0 e^{j\varphi_0} m_x\left(t + \frac{k}{2\pi}\right) e^{j\omega_0 t}\right\} = \mathrm{Re}\left\{A_0 e^{j\varphi_0} r\left(t + \frac{k}{2\pi}\right) e^{j\theta\left(t + \frac{k}{2\pi}\right)} e^{j\omega_0 t}\right\}$$

So, we obtain:

$y(t) = A_0 r\left(t + \dfrac{k}{2\pi}\right)\cos\left[\omega_0 t + \theta\left(t + \dfrac{k}{2\pi}\right) + \varphi_0\right]$. Introducing the "phase delay" $\tau_p = -\dfrac{\varphi(\omega_0)}{\omega_0}$ and the "group delay" $\tau_g = -\left.\dfrac{d\varphi}{d\omega}\right|_{\omega=\omega_0}$, we finally get:

$$y(t) = A_0 r\left(t - \tau_g\right)\cos\left[\omega_0(t - \tau_p) + \theta\left(t - \tau_g\right)\right]$$

We can remark that the carrier and the complex envelop are not delayed by the same amount (unless the phase response is a linear function of the frequency).