# EE411

# Principles of Communication

## 1. Introduction and Basic terminology of communication

In this course, we are going to learn the basic signal processing used in communication. But before doing so, we must have an appropriate understanding of what is the purpose of communication.
Communication can be defined as the process of transmitting "*reliably*" (or as reliably as possible) information from one point (*the source of information*) to another (*the destination*). This can be a transmission from one point to another distinct point (such as microwave link) or the source and destination can be located at the same place. This is what happens when a processor writes data in memory.
A typical block diagram of a communication system is shown in Figure 1- 1. The word "*Channel*" means the totality of the media and apparatus used for the transmission of the information. We will encounter this word quite often in this course and in future courses. It is a catch-all word used to describe many different things.

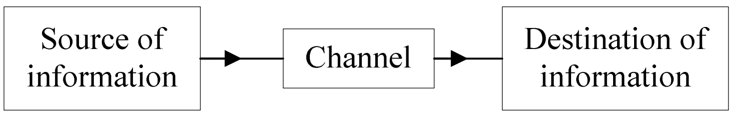


**Figure 1- 1 Block diagram of a communication system**

In information theory, this word (channel) means the mathematical mapping between the source and the destination. It is also used to describe the allocated band of frequencies given to a particular transmitting station. We will discover other meanings as we progress in the course. In many cases, we can use the word channel to describe the physical mean of transmission such as coaxial cables, fiber optic cables, etc.
The system shown in Figure 1- 1 is called a "*simplex*" communication system. In many cases, a transmitter and a receiver are located at the same place. If two such systems share the same communication mean, the system is called "*half-duplex*". In this case, we must have some kind of switching in order to connect the transmitter at one location with the receiver at the other location. An example of such system is the analogue telephone system. The handset contains a microphone and an earphone. This later is disconnected while the phone user is talking. There are also many "*push to talk*" systems that are half-duplex. On the other hand, if they use two different means, the system is called "*full-duplex*".


**Figure 1- 2 Simplex communication system**

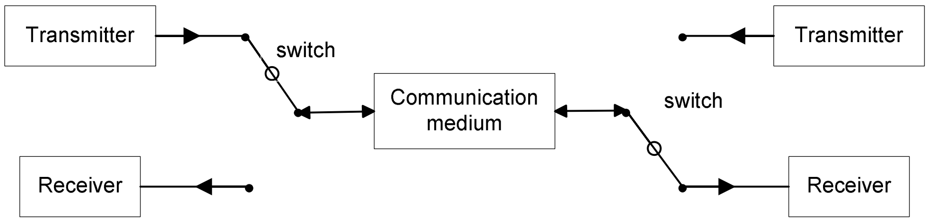


**Figure 1- 3 Half-duplex communication system**

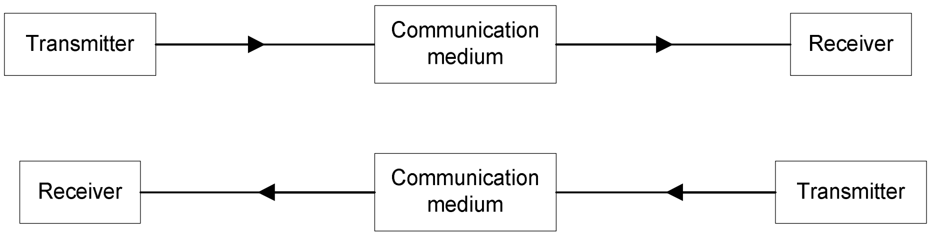


**Figure 1- 4 Full duplex communication system**

The above communication systems are point to point ones. There exist also "*broadcast*" systems where there is only one transmitter and many receivers. This is the case of the actual radio and television broadcast. Finally, there are communication systems where the transmitters and receivers are located at nodes of a network. The network can be local (LAN: *Local Area Network*) or it can cover a wide geographic region (WAN: *Wide Area Network* such as the *Internet*). Networks can be also classified according to the rules of connection. The telephone network connects two users for the whole duration of the communication while a *packet switching* network shares a communication link between many users (When the link is free, it is allocated to the users).

# 2. Review of signals and systems

## 2.1 Basic definitions

When we want to transmit information, we have to carry it using *signals*. Signals can represent the atmospheric pressure variations picked by a microphone in the case of speech communication. This signal is further transformed to an electrical signal. The microphone is the *transducer* that makes this transformation. Other type of signals can represent the light intensity variation in a black and white picture. So, we are going to call a "*signal*" a function that carries information. This function can be a mapping between a continuous variable (time) and a real or complex number.

$$x : t \in \mathbb{R} \mapsto x(t) \in \mathbb{C} \tag{0.1}$$

Equation (0.1) can represent a signal like the speech signal. A signal can also be represented by a sequence of real or complex numbers.

$$x : n \in \mathbb{Z} \mapsto x(n) \in \mathbb{C} \tag{0.2}$$

The above sequence can represent a signal generated inside a computer or a sequence of samples from an analogue signal. A black and white picture can be represented by a real valued function of two variables ($x$, $y$) that represent the position of the *pixel* in the image. In this course, we will consider essentially signals of the type represented by (0.1). Some important signals are:

The Heaviside unit step function:

$$u(t) = \begin{cases} 0 & t < 0 \\ \frac{1}{2} & t = 0 \\ 1 & t > 0 \end{cases} \tag{0.3}$$

The rotating phasor:

$$\phi(t) = A\exp\left(\omega_0 t + \theta\right) \tag{0.4}$$

This signal can be represented graphically in the complex plane by the rotating arrow shown in Figure 2- 1. $A$ is the amplitude (modulus) of the phasor, $\omega_0$ is the speed of rotation in radians/s and $\theta$ is the initial phase (at $t$ = 0). The sine wave signal is its projection on the real axis of the phasor.

$$x(t) = \mathrm{Re}\left[\phi(t)\right] = A\cos\left(\omega_0 t + \theta\right) \tag{0.5}$$

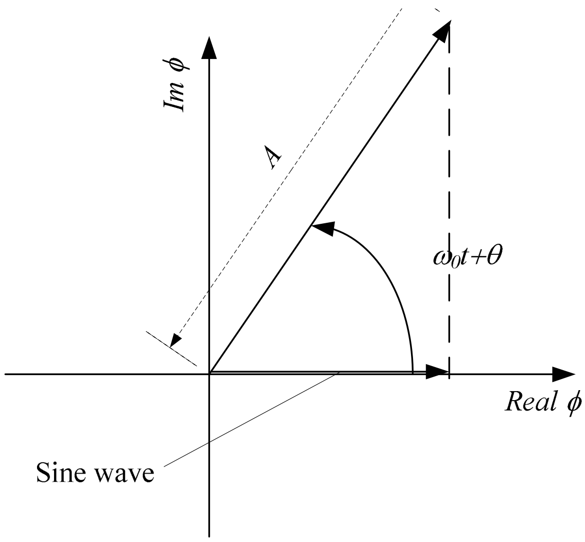


**Figure 2- 1 Rotating phasor and sine wave in the complex plane**

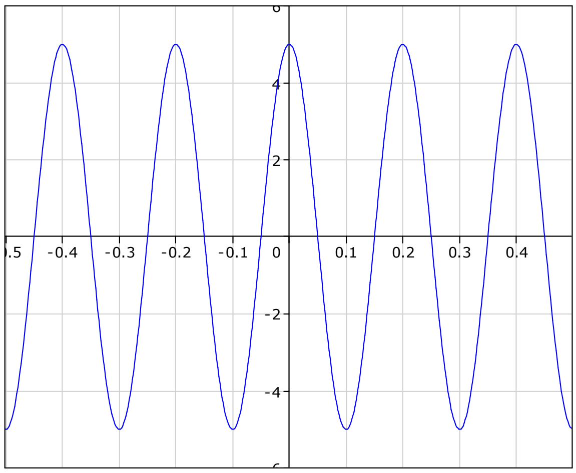


**Figure 2- 2 Sine wave in the time domain**

The sine wave is also related to the phasor by the following relation:

$$x(t) = \frac{\phi(t) + \phi^*(t)}{2} \tag{0.6}$$

This is represented graphically by Figure 2- 3. The signal $\phi^*(t)$ is the complex conjugate of the signal $\phi(t)$. It has the same amplitude, an opposite initial phase and it rotates in the opposite direction: we say that it has a negative frequency.

It is evident that three numbers are sufficient to characterize completely the phasor. They are: $\omega_0$, $A$ and $\theta$. We can represent this information graphically. Figure 2- 4 represents the "*spectrum*" of the phasor $\phi(t)$ or the single sided spectrum of the sine wave $x(t)$.

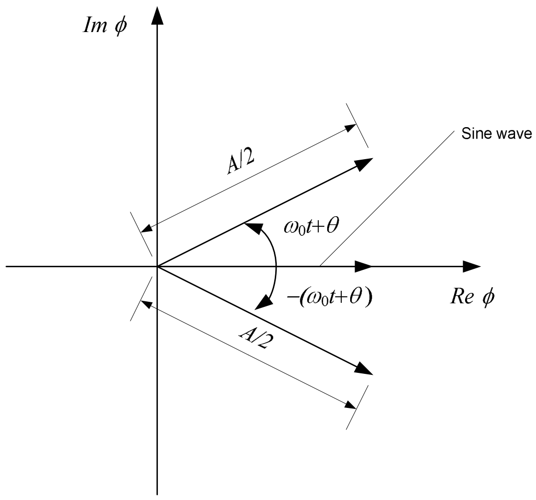


**Figure 2- 3**

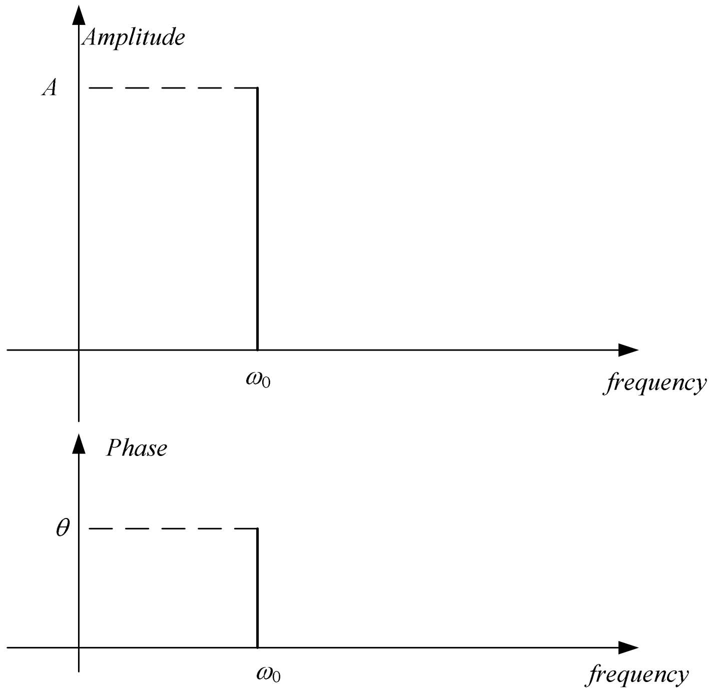


**Figure 2- 4 Spectrum of a phasor**

The above representation becomes interesting when a signal is the sum of many phasors.

$$x(t) = \sum_{k=1}^{N} A_k \exp\left(\omega_k t + \theta_k\right) \qquad (0.7)$$

The signal represented by (0.7) has $N$ different amplitudes $A_k$ and $N$ different phases $\theta_k$ located at the frequencies $\omega_k$.

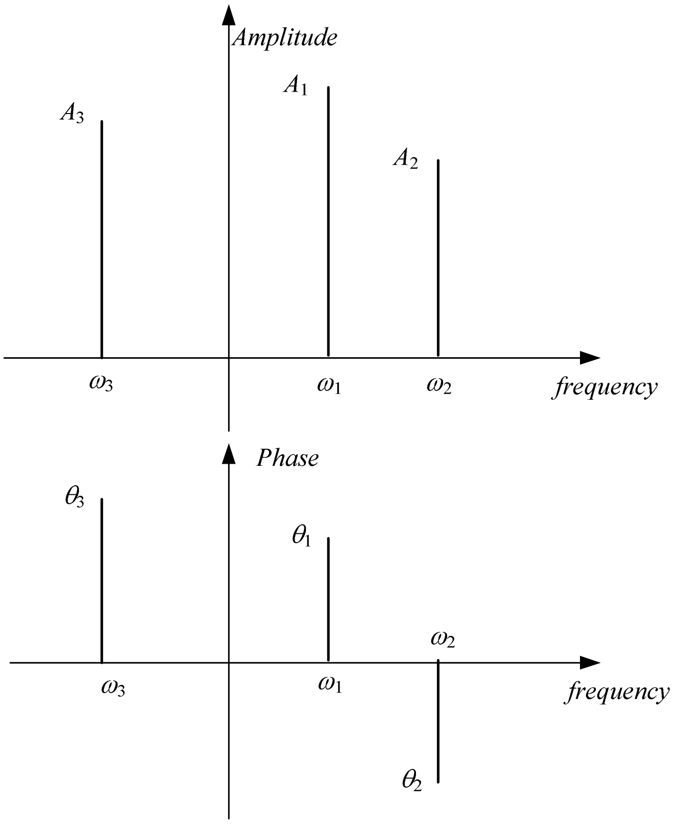


**Figure 2- 5 Two sided spectrum of the signal *x*(*t*)**

This representation of the signal $x(t)$ presupposes that the amplitudes $A_k$ and the phases $\theta_k$ are functions of the frequency $\omega$. In this case, for the signal $x(t)$ to be real, its amplitude spectrum must be an even function of the frequency and its phase spectrum must be an odd function of the frequency. In other words, if the amplitude has the value $A_1$ at the frequency $\omega = \omega_1$, it must have the same amplitude at the frequency $\omega = -\omega_1$ and if the phase has the value $\theta = \theta_1$ at the frequency $\omega = \omega_1$, the phase at $\omega = -\omega_1$ must be such that $\theta = -\theta_1$. An example is the spectrum of a sine wave $x(t) = A\cos(\omega_0 t + \theta)$ displayed in Figure 2- 6.

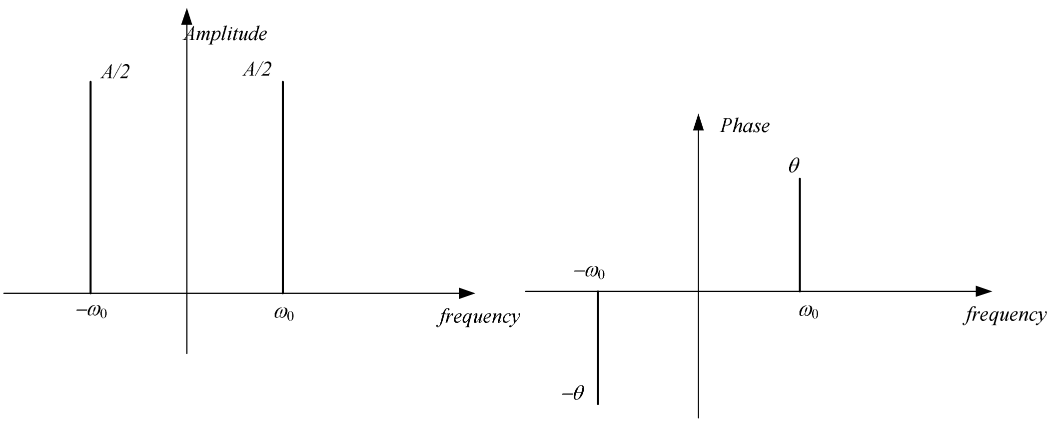


**Figure 2- 6 two sided spectrum of a sine wave**

It shows that the sine wave is the sum of two phasors of equal length rotating at the same speed but in opposite direction and starting with opposite initial phases.

## 2.2 Some signal properties

**Periodicity:**

We say that the signal $x(t)$ is periodic if it satisfies the following property:

$$\forall t \in \mathbb{R} \quad \exists T > 0 \text{ such that } x(t+T) = x(t)$$

The number $T$ is called the *period* of the signal. It is evident that if $T$ is a period of the signal then $kT$ ($k \in \mathbb{N}^*$) is also a period. For a periodic signal, there exist an infinite set of positive periods. The smallest one (we denote it $T_0$) is called the *fundamental period*. The phasor defined by equation (0.4) and the sine wave defined by equation (0.5) are two examples of periodic signals. Their fundamental period is:

$$T_0 = \frac{2\pi}{\omega_0} = \frac{1}{f_0} \tag{0.8}$$

and $f_0$ is the fundamental frequency (measured in Hertz). It indicates the number of turns per second (cycles/s) of the rotating phasor.

**Energy and power:**

If the signal $x(t)$ represents a current, then the instantaneous power that will be dissipated by a resistance $R$ will be $p(t) = Rx^2(t)$. In signal theory, we want a definition that will be independent on the value of $R$. So, we define the *instantaneous power* of the signal $x(t)$ as:

$$p(t) = |x(t)|^2 \tag{0.9}$$

i.e. it represents the power dissipated by a 1 Ω resistance if the signal corresponds to either current or voltage. We use the absolute value because the signal might be complex. At that time, the *energy* that will be dissipated in the interval of time $[t_1, t_2]$ is:

$$E[t_1,t_2] = \int_{t_1}^{t_2} p(t)dt = \int_{t_1}^{t_2} |x(t)|^2 \, dt \tag{0.10}$$

So, the total energy of the signal is:

$$E = \int_{-\infty}^{+\infty} |x(t)|^2 \, dt \tag{0.11}$$

Of course, the above integral must converge. If it diverges, then the signal will have an infinite energy. We can also define the average power in the interval $[-T/2, T/2]$ as:

$$P\left[-\frac{T}{2},\frac{T}{2}\right] = \frac{1}{T} E\left[-\frac{T}{2},\frac{T}{2}\right] = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} |x(t)|^2 \, dt \tag{0.12}$$

And the total average power (that will be called simply power) is the limit when $T$ goes to infinity of the above power.

$$P = \lim_{T\to+\infty} \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} |x(t)|^2 \, dt \tag{0.13}$$

The convergence properties of $E$ and $P$ impose a distinction or a classification of signals. If $E$ is finite, then $P$ will be zero. The signal is classified as an energy signal. However, for $P$ to be different from zero, we must have $E$ infinite. In this case, the signal is classified as a power signal.

**Energy signal:** $0 < E < \infty$ **;** $P = 0$ **.**

**Power signal:** $E = \infty$ **;** $0 < P < \infty$ **.**

Energy signals must decay to zero at $t = \pm\infty$. Any bounded finite time signal is then an energy signal.
For example, consider the signal $\Pi(t)$.

$$\Pi(t) = \begin{cases} 1 & -\frac{1}{2} \le t \le \frac{1}{2} \\ 0 & \text{elsewhere} \end{cases}$$

Its energy is of course $E = 1$. Consider now the following exponential signal.

$$x(t) = A\exp(-\alpha t)u(t) \quad ; \quad \alpha > 0$$

Its energy is: $E = \int_0^{+\infty} A^2 \exp(-2\alpha t) dt = -\frac{A^2}{2\alpha}\exp(-2\alpha t)\Big|_0^{+\infty} = \frac{A^2}{2\alpha}$

However, any periodic signal must be at least a power signal. In the case of periodic signals, the computation of the total average power is simplified if we use the periodicity. If we decompose the integral over $T$ as a sum of integrals over the fundamental period $T_0$, we obtain the following formula:

$$P = \frac{1}{T_0}\int_{T_0} |x(t)|^2 \, dt \qquad (0.14)$$

The notation $\int_{T_0}$ means an integral over any interval of length $T_0$, e.g. $[\alpha, \alpha+T_0[$.

Using (0.14), the power of the phasor is: $P = A^2$ and the power of the corresponding sine wave is: $P = A^2/2$.

### Time average

For power signals, we can define the time average of the signal (It is the dc value of the signal).

$$< x(t) >= \lim_{T \to +\infty} \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)dt \qquad (0.15)$$

For periodic signals, (0.15) reduces to

$$< x(t) >= \frac{1}{T_0}\int_{T_0} x(t)dt \qquad (0.16)$$

Time average can be defined for energy signals. However, the dc value of energy signals is zero.

## 2.3 Fourier series

The theory of Fourier series basically states that periodic signals can be expressed as a linear combination of phasors. We will see later that phasors form an essential set of signals and practically almost all signals of interest in communication are formed by such linear combinations. Let us consider a periodic signal $x(t)$ with fundamental period $T_0$. If this signal satisfies the following sufficient conditions:

**Dirichlet conditions:**

1. The signal $x(t)$ is bounded in the interval $[0, T_0]$.
2. It has a finite number of discontinuities in the interval $[0, T_0]$.
3. It has a finite number of extrema in the interval $[0, T_0]$.

Then it can be developed in the following series:

$$x(t) = \sum_{n=-\infty}^{+\infty} c_n \exp\left( jn\omega_0 t\right) \tag{0.17}$$

where $\omega_0 = 2\pi f_0 = {2\pi}/{T_0}$

We note that the Dirichlet conditions are sufficient and not necessary. This means that we can find functions that possess Fourier series without satisfying the above conditions. Even though the Fourier series have been defined for periodic signals, the development (0.17) can be applied to a finite time signal (i.e. a signal that is zero outside an interval of length $T_0$). In this case, the finite time signal will be equal to a periodic signal with period $T_0$ inside the interval. In this course, we are not going to study the convergence of the series. Simply, we can state its convergence at the discontinuities of $x(t)$.

Consider a signal $x(t)$ having a discontinuity at $t_0 \in [0, T_0]$. Let $x(t_0) = x_0$ and $\lim_{t\to t_0, t>t_0} x(t) = x(t_{0+})$; $\lim_{t\to t_0, t<t_0} x(t) = x(t_{0-})$ then:

$$\lim_{N\to+\infty} \sum_{n=-N}^{N} c_n \exp\left( jn\omega_0 t\right) = \frac{x(t_{0+}) + x(t_{0-})}{2} \tag{0.18}$$

Relation (0.18) explains why we have defined the value of the step function at the origin as $u(0) = 1/2$ and not 1 as it is usually defined in signal and system courses. The computation of the coefficients $c_n$ is quite instructive:

Let us multiply both sides of equation (0.17) by $\exp(-jk\omega_0 t)$ and integrate over $T_0$.

$$\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x(t)\exp\left(-jk\omega_0 t\right) dt = \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} \left\{ \sum_{n=-\infty}^{+\infty} c_n \exp\left( jn\omega_0 t\right)\right\} \exp\left(-jk\omega_0 t\right) dt$$

If the series converges, we can interchange the integration and the summation.

$$\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x(t)\exp\left(-jk\omega_0 t\right) dt = \sum_{n=-\infty}^{+\infty} c_n \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} \exp\left( jn\omega_0 t\right)\exp\left(-jk\omega_0 t\right) dt = \sum_{n=-\infty}^{+\infty} c_n \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} \exp\left( j\left(n-k\right)\omega_0 t\right) dt$$

The integral is not hard to evaluate and its value is:

$$\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} \exp\left[ j\left(n-k\right)\omega_0 t\right] dt = T_0 \frac{\sin \pi\left(n-k\right)}{\pi\left(n-k\right)} = T_0 \,\mathrm{sinc}\left(n-k\right) \tag{0.19}$$

Equation (0.19) defines the function sinc($x$) which is plotted below.

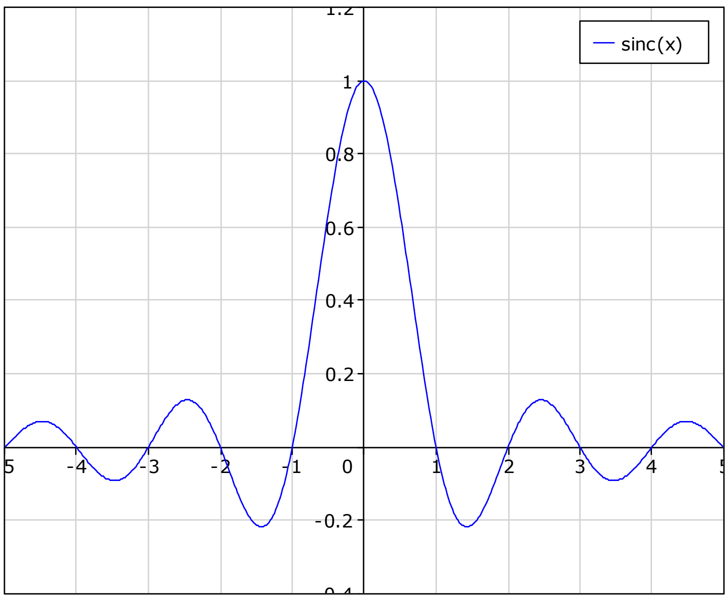


**Figure 2- 7 The sinc function**

This function has the following property:

$$\forall m \in \mathbb{Z} \quad ; \quad \mathrm{sinc}(m) = 1 \; if \;\; m = 0 \; and \; \mathrm{sinc}(m) = 0 \; if \;\; m \neq 0$$

Using the above property, we obtain:

$$\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x(t)\exp\left(-jk\omega_0 t\right)dt = \sum_{n=-\infty}^{+\infty} c_n T_0 \,\mathrm{sinc}(n-k) = c_k T_0$$

and finally, the coefficients of the Fourier series are given by:

$$c_n = \frac{1}{T_0}\int_{T_0} x(t)\exp\left(-jn\omega_0 t\right)dt \tag{0.20}$$

If the signal $x(t)$ is real, then all positive frequency phasors must be compensated by a negative frequency with same amplitude and opposite phase. In other words, we must have:

$$c_n = c_{-n}^* \tag{0.21}$$

or

$$\begin{aligned} |c_n| &= |c_{-n}| \\ \arg[c_n] &= -\arg[c_{-n}] \end{aligned} \tag{0.22}$$

In this case, we can use these symmetries to obtain the following alternate expressions of the Fourier series.

$$x(t) = c_0 + \sum_{n=1}^{\infty} 2|c_n|\cos\left(n\omega_0 t + \arg[c_n]\right) \tag{0.23}$$

By using the trigonometric identity: $\cos(a+b) = \cos a \cos b - \sin a \sin b$, we obtain the following result.

$$x(t) = c_0 + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t + \sum_{n=1}^{\infty} b_n \sin n\omega_0 t \tag{0.24}$$

where

$$a_n = |c_n|\cos\left(\arg[c_n]\right)$$

$$b_n = -|c_n|\sin\left(\arg\left[c_n\right]\right) \tag{0.25}$$

In (0.23), the term corresponding to $n = 1$ is called the fundamental, the terms corresponding to $n > 1$ are called the $n^{th}$ harmonics and of course $c_0$ is the average or dc value.

**Symmetry properties**

In the case of real signals, if the function $x(t)$ is even, i.e. $x(t) = x(-t)$, then the series (0.24) will not contain any sine term: $b_n = 0$ for all $n$. Alternatively, if the function is odd, i.e. $x(t) = -x(-t)$ then (0.24) will contain only sine terms, i.e. $c_0 = 0$ and $a_n = 0$.
If the real signal $x(t)$ has a half wave symmetry: $x(t \pm T_0/2) = -x(t)$, then its Fourier series will contain only odd harmonics, i.e. $c_n = 0$ for $n = 0, \pm 2, \pm 4, \ldots$

Example of Fourier series computation:

Consider the pulse train defined by: $x(t) = A\Pi\left(\frac{t}{\tau}\right)$ ; $0 < \tau < T_0$ for $-\frac{T_0}{2} \le t \le \frac{T_0}{2}$ and $x(t) = x(t + T_0)$ for all the other periods.

$$c_n = \frac{1}{T_0}\int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} A\exp\left(-jn\omega_0 t\right)dt = \frac{1}{T_0}\frac{A}{-jn\omega_0}\exp\left(-jn\omega_0 t\right)\Big|_{-\frac{\tau}{2}}^{\frac{\tau}{2}}$$

$$= A\frac{\tau}{T_0}\operatorname{sinc}\left(\frac{n\tau}{T_0}\right)$$

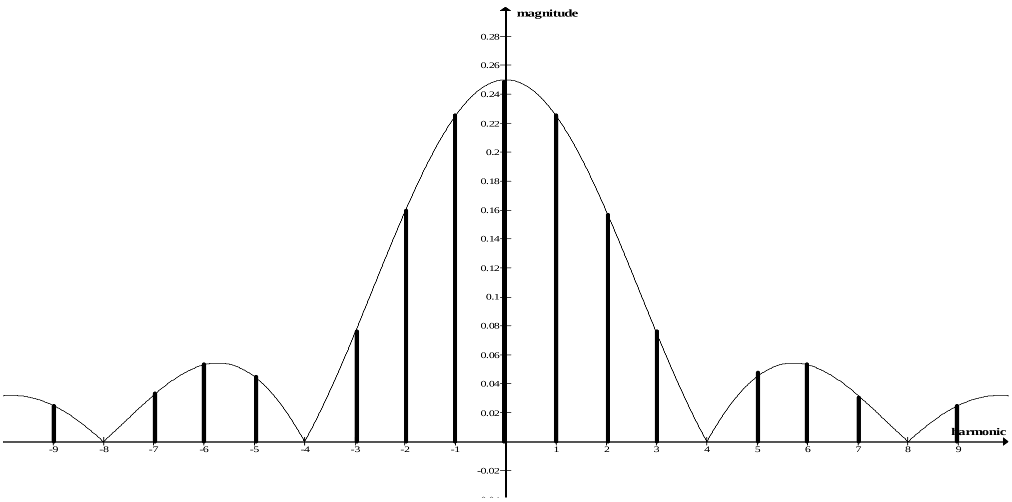


**Figure 2- 8 Amplitude spectrum of the pulse train**

The above spectrum corresponds to harmonics located at frequencies $n\omega_0 = n\frac{2\pi}{T_0}$ along with a duty cycle $d = \frac{\tau}{T_0}$.

If now $\tau = T_0/2$ (square wave), we obtain: $c_n = \frac{A}{2}\operatorname{sinc}\left(\frac{n}{2}\right)$. This means that $c_n = \frac{A}{\pi n}\sin\frac{n\pi}{2}$, giving

$$c_n = \begin{cases} \dfrac{A}{2} & n=0 \\ 0 & n \text{ even} \\ \dfrac{A}{\pi}, -\dfrac{A}{3\pi}, \dfrac{A}{5\pi}, \ldots & n = \pm1, \pm3, \pm5, \ldots \end{cases}$$

Grouping the terms corresponding to + and – $n$, we obtain the following series:

$$x(t) = \frac{A}{2} + \frac{2A}{\pi}\left( \cos\omega_0 t - \frac{1}{3}\cos\omega_0 t + \frac{1}{5}\cos 5\omega_0 t - \cdots \right)$$

**Parseval's relation:**

Let us consider two signals $x(t)$ and $y(t)$, both periodic with period $T_0$. Their Fourier development is:

$$x(t) = \sum_{n=-\infty}^{+\infty} X_n \exp\left( j\omega_0 t \right)$$

$$y(t) = \sum_{n=-\infty}^{+\infty} Y_n \exp\left( j\omega_0 t \right)$$

Then

$$\frac{1}{T_0}\int_{T_0} x(t) y^*(t) dt = \sum_{n=-\infty}^{+\infty} X_n Y_n^* \qquad (0.26)$$

Proof:

$$\frac{1}{T_0}\int_{T_0} x(t) y^*(t) dt = \frac{1}{T_0}\int_{T_0} \left\{ \sum_{n=-\infty}^{+\infty} X_n \exp\left( jn\omega_0 t \right) \sum_{m=-\infty}^{+\infty} Y_m^* \exp\left( -jm\omega_0 t \right) \right\} dt$$

We interchange the integral with the sums.

$$\frac{1}{T_0}\int_{T_0} x(t) y^*(t) dt = \sum_{n=-\infty}^{+\infty} \sum_{m=-\infty}^{+\infty} X_n Y_m^* \left\{ \frac{1}{T_0}\int_{T_0} \exp\left( j(n-m)\omega_0 t \right) dt \right\}$$

Using the properties of the sinc function, we finally obtain:

$$\frac{1}{T_0}\int_{T_0} x(t) y^*(t) dt = \sum_{n=-\infty}^{+\infty} X_n Y_n^*$$

(q.e.d.)

Relation (0.26) can be applied for the case $y(t) = x(t)$. We obtain

$$P = \frac{1}{T_0}\int_{T_0} \left| x(t) \right|^2 dt = \sum_{-\infty}^{+\infty} \left| X_n \right|^2 \qquad (0.27)$$

In other words, (0.27) means that the total average power of the waveform $x(t)$ is the sum of the powers of all the phasors constituting $x(t)$.