EE174: Make-up solutions

(1) a) Since $S \subset \mathbb{R}^3(\mathbb{R})$, we need only show it is closed under L.C.

Let $\begin{cases} v_1, v_2 \in S \\ \alpha_1, \alpha_2 \in \mathbb{R} \end{cases} \Rightarrow \alpha_1 v_1 + \alpha_2 v_2 = \alpha_1(x_1, y_1, z_1) + \alpha_2(x_2, y_2, z_2) = (\underbrace{\alpha_1 x_1 + \alpha_2 x_2}_{\bar{x}}, \underbrace{\alpha_1 y_1 + \alpha_2 y_2}_{\bar{y}}, \underbrace{\alpha_1 z_1 + \alpha_2 z_2}_{\bar{z}})$

Now $\bar{x} + 2\bar{y} + \bar{z} = (\alpha_1 x_1 + \alpha_2 x_2) + 2(\alpha_1 y_1 + \alpha_2 y_2) + (\alpha_1 z_1 + \alpha_2 z_2) = \alpha_1 \underbrace{(x_1 + 2y_1 + z_1)}_{=0} + \alpha_2 \underbrace{(x_2 + 2y_2 + z_2)}_{=0} = 0$

Hence S is a subspace of $\mathbb{R}^3(\mathbb{R})$.

b). S has 2 degrees of freedom $\Rightarrow \dim S = 2 \Rightarrow S$ is a plane with equation $x + 2y + z = 0$

(2) a) $r(A) = 2$ since 2 l.i. columns $\Rightarrow$ a basis for $R(A)$ is $\left\{ \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}; \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\}$.

$n(A) = 3 - 2 = 1 \Rightarrow$. A basis for $N(A)$ is any nonzero vector x s.t. $Ax = \theta$.

$$\begin{pmatrix} 1 & 0 & 1 \\ -1 & 1 & 1 \\ 1 & 1 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} x + z = 0 \\ -x + y + z = 0 \\ x + y + 3z = 0 \end{cases} \Rightarrow \begin{matrix} x = -z \\ y = -2z \end{matrix} \Rightarrow x = \begin{pmatrix} -z \\ -2z \\ z \end{pmatrix} = z \begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix}$$

Hence $N(A) = \left[\begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix} \right]$ and a basis is $\left\{ \begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix} \right\}$

b) $Y \in R(A) \Rightarrow Y$ can be written uniquely in basis of $R(A)$:

$\exists \alpha_1, \alpha_2$ s.t. $Y = \alpha_1 A_1 + \alpha_2 A_2$ or $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \alpha_1 \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} + \alpha_2 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \Rightarrow \begin{cases} \alpha_1 = 1 \\ -\alpha_1 + \alpha_2 = 0 \\ \alpha_1 + \alpha_2 = 1 \end{cases}$

leads to $\alpha_1 = 1$ and $\alpha_1 = 1/2$ which is impossible $\Rightarrow Y \notin R(A)$.

(3) a) $\begin{cases} T(e_1) = 7e_1 - 6e_2 \\ T(e_2) = 12e_1 - 10e_2 \end{cases}$ and $\begin{cases} v_1 = 3e_1 - 2e_2 \\ v_2 = 4e_1 - 3e_2 \end{cases} \Rightarrow \begin{matrix} T(v_1) = 3T(e_1) - 2T(e_2) = -3e_1 + 2e_2 \\ T(v_2) = 4T(e_1) - 3T(e_2) = -8e_1 + 6e_2 \end{matrix}$

Now $\begin{cases} e_1 = 3v_1 - 2v_2 \\ e_2 = 4v_1 - 3v_2 \end{cases} \Rightarrow \begin{cases} T(v_1) = -v_1 \\ T(v_2) = -2v_2 \end{cases} \Rightarrow B = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix}$

b) Clearly B is diagonal: with this choice we obtain a diagonal representation.

(4) a) $AA^T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I$

b) Clearly A is nonsingular by definition and $A^{-1} = A^T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix}$

(5) $\det(A) = |A| = (x-1) \begin{vmatrix} 2 & 1 \\ x-2 & 1 \end{vmatrix} - \begin{vmatrix} 1 & x+2 \\ x-2 & 1 \end{vmatrix} = 5x - 9$

Hence A singular $\Leftrightarrow \det(A) = 0 \Leftrightarrow \boxed{x = 9/5}$